

NORMAL FUNCTIONS, PICARD-FUCHS EQUATIONS, AND ELLIPTIC FIBRATIONS ON K3 SURFACES

XI CHEN[†], CHARLES DORAN[†], MATT KERR^{††}, AND JAMES D. LEWIS[†]

ABSTRACT. Using Gauss-Manin derivatives of normal functions, we arrive at some remarkable results on the non-triviality of the transcendental regulator for K_m of a very general projective algebraic manifold. Our strongest results are for the transcendental regulator for K_1 of a very general $K3$ surface. We also construct an explicit family of K_1 cycles on $H \oplus E_8 \oplus E_8$ -polarized $K3$ surfaces, and show they are indecomposable by a direct evaluation of the real regulator. Critical use is made of natural elliptic fibrations, hypersurface normal forms, and an explicit parametrization by modular functions.

CONTENTS

1. The main results	2
2. Derivatives of normal functions I	5
2.1. Gauss-Manin derivatives	5
2.2. Nontriviality of transcendental regulators	6
3. Derivatives of normal functions II	8
3.1. Picard-Fuchs equations associated to regulators	8
3.2. Nontriviality of Picard-Fuchs operators	9
4. Proof of Theorem 1.3	11
4.1. Non-triviality of $\Phi_{3,1}$	12
4.2. The purely transcendental regulator $\Psi_{3,1}$	13
5. Intermezzo: Lattice polarized $K3$ surfaces, hypersurface normal forms, and modular parametrization	14
5.1. Normal forms and elliptic fibrations	15
5.2. Picard-Fuchs equations and modular parametrization	17
6. Explicit K_1 class on a family of Shioda-Inose $K3$ surfaces	20
6.1. Kummer $K3$ geometry	20
6.2. Normalization of $\widetilde{\mathcal{E}}'_1$	23
6.3. The $(1,1)$ current	24
6.4. The pairing	25
6.5. Interpretation of the integrals	26
Appendix to Section 3	28
7. The transcendental regulator for a Picard rank 20 $K3$	29

Date: September 2, 2011.

1991 *Mathematics Subject Classification.* Primary 14C25; Secondary 14C30, 14C35.

Key words and phrases. Regulator, Chow group, Deligne cohomology, Picard-Fuchs equation, normal function, $K3$ surface, elliptic fibration, normal form, lattice polarization.

[†] Partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada. ^{††} Partially supported by NSF grant DMS-1068974.

1. THE MAIN RESULTS

Let X be a projective algebraic manifold of dimension d , and $\mathrm{CH}^r(X, m)$ the higher Chow group introduced by Bloch ([B]). We are mainly interested in working modulo torsion, thus we will restrict ourselves to the corresponding group $\mathrm{CH}^r(X, m; \mathbb{Q}) := \mathrm{CH}^r(X, m) \otimes \mathbb{Q}$. An explicit description of the Bloch cycle class map to Deligne cohomology,

$$\mathrm{cl}_{r,m} : \mathrm{CH}_{\mathrm{hom}}^r(X, m; \mathbb{Q}) \rightarrow J(H^{2r-m-1}(X, \mathbb{Q}(r))) \subset H_{\mathcal{D}}^{2r-m}(X, \mathbb{Q}(r)),$$

is given in [KLM], where

$$\begin{aligned} J(H^{2r-m-1}(X, \mathbb{Q}(r))) &:= \mathrm{Ext}_{\mathrm{MHS}}^1(\mathbb{Q}(0), H^{2r-m-1}(X, \mathbb{Q}(r))) \\ &\simeq \frac{F^{d-r+1} H^{2d-2r+m+1}(X, \mathbb{C})^\vee}{H_{2d-2r+m+1}(X, \mathbb{Q}(d-r))}. \end{aligned}$$

We will now assume that X is a very general member of a family $\lambda : \mathcal{X} \rightarrow \mathcal{S}$, where \mathcal{X}, \mathcal{S} are smooth quasi-projective varieties and λ is smooth and proper, and where $X := \lambda^{-1}(0)$ corresponds to $0 \in \mathcal{S}$. Associated to this is the Kodaira-Spencer map $\kappa : T_0(\mathcal{S}) \rightarrow H^1(X, \Theta_X)$, whose image we will denote by $H_{\mathrm{alg}}^1(X, \Theta_X)$, where Θ_X is the sheaf of holomorphic vector fields on X .

The Hodge structure

$$H^{2r-m-1}(X, \mathbb{Q}(r)) = H_f^{2r-m-1}(X, \mathbb{Q}(r)) \bigoplus H_v^{2r-m-1}(X, \mathbb{Q}(r))$$

decomposes, where

$$H_f^{2r-m-1}(X, \mathbb{Q}(r)) := H^{2r-m-1}(X, \mathbb{Q}(r))^{\pi_1(\mathcal{S})}$$

is the fixed part of the corresponding monodromy group action on $H^{2r-m-1}(X, \mathbb{Q}(r))$, and $H_v^{2r-m-1}(X, \mathbb{Q}(r))$ is the orthogonal complement.

One has a reduced cycle class map

$$\underline{\mathrm{cl}}_{r,m} : \mathrm{CH}^r(X, m; \mathbb{Q}) \rightarrow J(H_v^{2r-m-1}(X, \mathbb{Q}(r))).$$

Such a regulator plays a key role in detecting interesting $\mathrm{CH}^r(X, m)$ classes, such as indecomposables (see for example [L] and [MS]). We can further pass to the transcendental regulator

$$\Phi_{r,m} : \mathrm{CH}^r(X, m; \mathbb{Q}) \rightarrow \frac{F^{d-r+m+1} H_v^{2d-2r+m+1}(X, \mathbb{C})^\vee}{H_{2d-2r+m+1}^v(X, \mathbb{Q}(d-r))},$$

for which the formula is given as follows. If $\xi \in \mathrm{CH}^r(X, m; \mathbb{Q})$, then viewing $|\xi| \subset X \times \mathbb{C}^m$ as a closed subset of codimension r , it follows that $\dim \mathrm{Pr}_X(|\xi|) \leq d - r + m$. According to the formula in [KLM] (also cf. [K-L]), one can choose ξ such that it meets the real cube $X \times [-\infty, 0]^m$ properly, and that for $\omega \in F^{d-r+m+1} H_v^{2d-2r+m+1}(X, \mathbb{C})$,

$$\Phi_{r,m}(\xi)(\omega) = \pm \frac{1}{(2\pi i)^{d-r}} \int_{\partial^{-1}\{\xi \cap X \times [-\infty, 0]^m\}} \omega.$$

For example in the case $(d, r, m) = (2, 2, 1)$, we have

$$\Phi_{2,1} : \mathrm{CH}^2(X, 1) \rightarrow \frac{H_v^{2,0}(X)^\vee}{H_2^v(X, \mathbb{Q})}.$$

Our first result is the following.

Theorem 1.1. *Let X/\mathbb{C} be a very general algebraic K3 surface. Then the transcendental regulator $\Phi_{2,1}$ is non-trivial. Quite generally, if X is a very general member of a general subfamily of dimension $20 - \ell$, describing a family of K3 surfaces with general member of Picard rank ℓ , with $\ell < 20$, then $\Phi_{2,1}$ is non-trivial.*

From the proof of Theorem 1.1, we deduce:

Corollary 1.1. *Let X/\mathbb{C} be a very general member of a family of surfaces for which $H_{\mathrm{alg}}^1(X, \Theta_X) \otimes H_v^{2,0}(X) \rightarrow H_v^{1,1}(X)$ is surjective. If the real regulator $r_{2,1} : \mathrm{CH}^2(X, 1) \rightarrow H_v^{1,1}(X, \mathbb{R}(1))$ is non-trivial, then so is the transcendental regulator $\Phi_{2,1}$.*

Now consider X of dimension d as a very general member of a family $\lambda : \mathcal{X} \rightarrow \mathcal{S}$. With a little bit of effort, one can also show the following.

Theorem 1.2. *Suppose that the cup product induced map*

$$H_{\mathrm{alg}}^1(X, \Theta_X) \otimes H_v^{d-r+m+1-\ell, d-r+\ell}(X) \rightarrow H_v^{d-r+m-\ell, d-r+\ell+1}(X),$$

is surjective for all $\ell = 0, \dots, m-1$. Then $\underline{\mathrm{cl}}_{r,m} \neq 0 \Rightarrow \Phi_{r,m} \neq 0$.

Theorems 1.1, 1.2 and Corollary 1.1 will be proved in section 2. We deduce from Theorem 1.2 the following:

Corollary 1.2. *Let X be a very general K3 surface, and $H_v^2(X, \mathbb{C})$ be transcendental cohomology. Then the transcendental regulator*

$$\Phi_{3,1} : \mathrm{CH}^3(X \times X, 1) \rightarrow \frac{\{F^3(H_v^2(X, \mathbb{C}) \otimes H_v^2(X, \mathbb{C}))\}^\vee}{H_4(X \times X, \mathbb{Q}(1))},$$

is non-trivial.

We prove Corollary 1.2 in section 4. It turns out however, that with more effort, we can actually prove the following stronger result:

Theorem 1.3. *The purely transcendental regulator*

$$\Psi_{3,1} : \mathrm{CH}^3(X \times X, 1) \rightarrow \frac{H^{4,0}(X \times X, \mathbb{C})^\vee}{H_4(X \times X, \mathbb{Q}(1))},$$

is non-trivial for a very general K3 surface X .

The proofs of all the above results rely on a very simple trick involving the infinitesimal invariant of a normal function associated to a family of cycles on \mathcal{X}/\mathcal{S} inducing a given transcendental regulator value on X . A deeper question asks whether such a normal function is detected by a Picard-Fuchs operator. A blanket answer to this question is a yes; however rather than explain it here, we provide a complete clarification in §3.

Now returning to Theorem 1.1, two questions come to mind. First, the method of [C-L1], which proves the existence of deformations of decomposables on Picard rank 20 K3's, to indecomposables on a general polarized K3, is highly non-explicit.

How can one construct interesting *explicit* examples of cycles with nontrivial $\Phi_{2,1}$ on subfamilies with $\ell > 1$? Second, on a Picard-rank 20 $K3$, does one *expect* there to be any cycles at all which have nontrivial $\Phi_{2,1}$, and which are therefore indecomposable?

The first question is our main concern for the remainder of the paper. In §5, we introduce a crucial set of tools needed for explicit computations in this setting. The notion of a polarized $K3$ surface is extended to that of a lattice polarization, and algebraic hypersurface normal forms are given for certain families of lattice polarized $K3$ surfaces of high Picard rank ℓ . We then describe a very useful “internal structure” consisting of an elliptic fibration with section(s). Explicit Picard-Fuchs operators are given and related to parametrizations of coarse moduli spaces by modular functions and their generalizations.

Starting in §6, we restrict our considerations to $\ell = 18$ or 19 , where there have been a number of ideas that have not panned out. The article by [PL-MS], which in itself is an interesting piece of work, considers a cycle \mathcal{Z} on a 1-parameter family of elliptically fibered $K3$ ’s with $\ell = 19$ and a choice of section ω of the relative canonical bundle. In this context $F := \Phi_{2,1}(\mathcal{Z})(\omega)$ is a multivalued holomorphic function and the indecomposability of \mathcal{Z} may be detected by showing the Picard-Fuchs operator for ω does not annihilate it. Unfortunately, this cycle turns out to be 2-torsion,¹ and the computation of F leaves out a part of the membrane integral which cancels the part written down. For $\ell = 18$, one can try to construct regulator-indecomposable cycles on a product $E_1 \times E_2$ of elliptic curves and then pass to the Kummer. Such a construction is attempted in [G-L] but this cycle, too, was shown by M. Saito to be decomposable.² When $E_1 \cong E_2$, other authors (cf. [Zi]) have investigated “triangle cycles” supported on $E \times \{p\}$, $\{q\} \times E$, and the diagonal Δ_E , where $[p] - [q]$ is N -torsion. But this cannot produce indecomposable cycles, since the sum of the natural N^2 N -torsion translates of such a cycle (by integer multiples of $p - q$ on the two factors) is both visibly decomposable and (up to torsion) equivalent to N^2 times the original cycle.

With this discouraging history, it is easy to imagine that when X is an elliptically fibered $K3$, the very natural $\mathrm{CH}^2(X, 1)$ classes supported on semistable singular (Kodaira type I_n) fibers might be decomposable as well. Indeed one knows in the case of a modular elliptic fibration ($K3$ or not), that Beilinson’s Eisenstein symbols [Be] kill all such classes. On the other hand, using arithmetic methods to bound the rank of the dlog image, Asakura [As] demonstrated that for elliptic surfaces with general fiber $y^2 = x^3 + x^2 + t^n$ ($n \in [7, 29]$ prime), the type I_1 fibers generate $n - 1$ independent indecomposable K_1 classes. His paper stops short of attempting any regulator computations for such cycles, and this is what we take up in §3 in the context where the surface and cycle are allowed to vary.

Specifically, using an I_1 fiber in an internal elliptic fibration of the 2-parameter family $\{X_{a,b}\}$ of Shioda-Inose $K3$ ’s ($\ell = 18$) [C-D2], we write down a (multivalued) family of cycles $\mathcal{Z}_{a,b} \in \mathrm{CH}^2(X_{a,b}, 1)$. Passing to the associated Kummer family with parameters α, β (and cycle $\mathcal{Z}_{\alpha,\beta}$), we find that the family of cycles becomes single-valued over the diagonal ($\ell = 19$) sublocus $\alpha = \beta$, which is the Legendre modular curve $\mathbb{P}^1 \setminus \{0, 1, \infty\} \cong \mathfrak{H}/\Gamma(2)$. At this point we write down a smooth family of real

¹the cycle, which is supported over $\{Z = 0\} \cup \{Z = 1\} \cup \{X = 0\} \cup \{X = \infty\}$ in the notation of [PL-MS], is in fact one-half the residue of the symbol $\{X, 1 - \frac{1}{Z}\}$.

²that construction can, however, be corrected [T].

closed $(1, 1)$ forms η_α and compute directly the function

$$\psi(\alpha) := r_{2,1}(Z_{\alpha,\alpha})(\eta_\alpha) = -8|\alpha + 1| \operatorname{Im} \int_{\mathbb{C}} z \cdot \log \left| \frac{z+i}{z-i} \right| \left\{ \frac{\{(\alpha^2 - \alpha - 1)z^4 + 2z^2 + (\alpha^3 - \alpha^2 - 2\alpha + 1)\}}{|z^2 - \alpha||1 - \alpha z^2||z^2 + 1||z^2 - (1 + \alpha - \alpha^2)|} \times \frac{\{(\alpha^3 - \alpha^2 - 2\alpha + 1)z^4 + 2z^2 + (\alpha^2 - \alpha - 1)\}}{|(1 + \alpha - \alpha^2)z^2 - 1||z^4 + (\alpha^3 - 3\alpha)z^2 + 1|} \right\} dx \wedge dy$$

to be nonzero. By Corollary 1.3 we have immediately the

Theorem 1.4. $\Phi_{2,1}(\mathcal{Z}_{a,b})$ is non-trivial for very general (a, b) , and $\mathcal{Z}_{a,b}$ is indecomposable.

Remark 1.1. In fact, $\psi(\alpha)$ is a very interesting function. Dividing out by the volume of the Legendre elliptic curve and pulling back by the classical modular function λ to obtain a function $\tilde{\psi}(\tau)$ on \mathfrak{H} , yields a “Maass cusp form with pole”. That is, $\tilde{\psi}$ is $\Gamma(2)$ -invariant, is smooth away from the λ -preimage of $\alpha = -1$ (where it has a $\log|\cdot|$ singularity), dies at the 3 cusps, and (away from these bad points) is an eigenfunction of the hyperbolic Laplacian $-y^2\Delta$. This will be shown in a follow-up paper of the third author [Ke].

Finally, we turn briefly to the second question, concerning the case $\ell = 20$, in §7. Due to the vanishing of $H_v^{1,1}(X, \mathbb{R})$, $r_{2,1}$ is zero by definition, but this is no reason for the transcendental Abel Jacobi map $\Phi_{2,1}$ to vanish. In the example we work out, whether or not $\Phi_{2,1}(\mathcal{Z})$ is nontorsion boils down to the irrationality of a single number (cf. (103)), which we do not know how to prove directly. It seems likely both that the cycle is indecomposable and that this may be shown by using the methods in [As] to compute the dlog image.

2. DERIVATIVES OF NORMAL FUNCTIONS I

2.1. Gauss-Manin derivatives. We first prove Theorem 1.1 and Corollary 1.1, after which the general argument pertaining to Theorem 1.2 will follow rather easily. Consider a smooth family $\pi : \mathcal{X} \rightarrow S$ of $K3$ surfaces polarized by a relatively ample line bundle L over a polydisk S , with central fiber X . We have the Gauss-Manin (GM) connection:

$$(1) \quad \nabla : \mathcal{O}_S \otimes R^q \pi_* \mathbb{C} \rightarrow \Omega_S^1 \otimes R^q \pi_* \mathbb{C}$$

which is a flat connection that $\nabla^2 = 0$ and satisfies the Griffiths transversality:

$$\nabla(\mathcal{O}_S \otimes F^p R^q \pi_* \mathbb{C}) \subset \Omega_S^1 \otimes F^{p-1} R^q \pi_* \mathbb{C}.$$

Let Θ_S be the holomorphic tangent bundle of S . We can think of Θ_S as the sheaf of holomorphic linear differential operators. By identifying $\partial/\partial z_k$ with $\nabla_{\partial/\partial z_k}$, Θ_S acts on $\mathcal{O}_S \otimes R^q \pi_* \mathbb{C}$ via

$$(2) \quad u \cdot \omega = \nabla_u \omega$$

for $u \in H^0(\Theta_S)$, where we write $H^0(-)$ for $H^0(S, -)$.

We fix a nonzero section $\omega \in H^0(K_{\mathcal{X}/S})$. For all $u \in H^0(\Theta_S)$ and all $\gamma \in H^2(X, \mathbb{C})$ (where $H^2(X, \mathbb{C})$ is identified with $H^0(S, R^2 \pi_* \mathbb{C})$, using S a polydisk),

$$(3) \quad u \langle \gamma, \omega \rangle = \langle \gamma, \nabla_u \omega \rangle.$$

Let $\xi \in \text{CH}^2(\mathcal{X}/S, 1)$ be the result of an algebraic deformation of a cycle in the central fiber X restricted to \mathcal{X}/S , and $\text{cl}_{2,1}$ be the regulator map

$$(4) \quad \text{cl}_{2,1} : \text{CH}^2(\mathcal{X}/S, 1) \rightarrow H^0\left(\frac{\mathcal{O}_S \otimes R^2\pi_*\mathbb{C}}{\mathcal{O}_S \otimes F^2R^2\pi_*\mathbb{C} + R^2\pi_*\mathbb{Q}(2)}\right).$$

We let ν be a lift of $\text{cl}_{2,1}(\xi)$ to $H^0(\mathcal{O}_S \otimes R^2\pi_*\mathbb{C})$. We know that $\langle \nabla_u \nu, \omega \rangle = 0$ since the map

$$(5) \quad \begin{aligned} \nabla \circ \text{cl}_{2,1} : \text{CH}^2(\mathcal{X}/S, 1) &\xrightarrow{\text{cl}_{2,1}} H^0\left(\frac{\mathcal{O}_S \otimes R^2\pi_*\mathbb{C}}{\mathcal{O}_S \otimes F^2R^2\pi_*\mathbb{C} + R^2\pi_*\mathbb{Q}(2)}\right) \\ &\xrightarrow{\nabla} H^0\left(\Omega_S^1 \otimes \frac{R^2\pi_*\mathbb{C}}{F^1R^2\pi_*\mathbb{C}}\right) \end{aligned}$$

induced by the GM connection is trivial. This follows from the horizontality condition on normal functions associated to (higher Chow) algebraic cycles - well known among experts, the horizontality condition (leading to the infinitesimal invariant of normal functions) being deducible for example from a Deligne cohomology spectral sequence argument in [C-MS-P] (pp. 267-269), and adapted to higher Chow cycles. [For the convenience of the reader, here is how the argument works. Recall the analytic Deligne complex $0 \rightarrow \mathbb{Z}(r) \rightarrow \Omega_X^{\bullet < r}$, which leads to an exact sequence $\mathbb{H}^{2r-m-1}(\Omega_X^{\bullet < r}) \rightarrow H_{\mathcal{D}}^{2r-m}(\mathcal{X}, \mathbb{Z}(r)) \rightarrow H^{2r-m}(\mathcal{X}, \mathbb{Z}(r))$. We consider a null-homologous cycle in $\text{CH}^r(X, m)$ that spreads to a (relatively null-homologous) cycle on $\text{CH}^r(\mathcal{X}/S, m)$, which will map to zero in $H^{2r-m}(\mathcal{X}, \mathbb{Z}(r))$ (as S is a polydisk), and hence the induced normal function has a lift in $\mathbb{H}^{2r-m-1}(\Omega_X^{\bullet < r})$. The Leray spectral sequence gives us an edge map $\mathbb{H}^{2r-m-1}(\Omega_X^{\bullet < r}) \rightarrow H^0(S, \mathbb{R}^{2r-m-1}\pi_*\Omega_X^{\bullet < r})$. One has a filtering of the complex $\mathcal{L}^\nu \Omega_X^{\bullet < r} := \text{Image}(\pi^*\Omega_S^\nu \otimes \Omega_X^{\bullet < r-\nu} \rightarrow \Omega_X^{\bullet < r})$, with $Gr_{\mathcal{L}}^\nu = \pi^*\Omega_S^\nu \otimes \Omega_{X/S}^{\bullet < r-\nu} \simeq \Omega_S^\nu \otimes \Omega_{X/S}^{\bullet < r-\nu}$. There is a spectral sequence computing $\mathbb{R}^{p+q}\pi_*\Omega_X^{\bullet < r}$ with $\mathcal{E}_1^{p,q} = \mathbb{R}^{p+q}Gr_{\mathcal{L}}^p = \Omega_S^p \otimes \mathbb{R}^q\rho_*\Omega_{X/S}^{\bullet < r-p}$. So we have the composite $H^0(S, \mathbb{R}^{2r-m-1}\pi_*\Omega_X^{\bullet < r}) \rightarrow H^0(S, \mathcal{E}_1^{0,2r-m-1}) \xrightarrow{d_1} H^0(S, \mathcal{E}_1^{1,2r-m-1})$, which must be zero by spectral sequence degeneration, using the fact that $\mathcal{E}_\infty^{0,2r-m-1} \subset \ker(d_1 : \mathcal{E}_1^{0,2r-m-1} \rightarrow \mathcal{E}_1^{1,2r-m-1})$. But $H^0(S, \mathcal{E}_1^{0,2r-m-1}) \xrightarrow{d_1} H^0(S, \mathcal{E}_1^{1,2r-m-1})$ is precisely the Gauss-Manin connection

$$H^0(S, \mathbb{R}^{2r-m-1}\pi_*\Omega_{X/S}^{\bullet < r}) \xrightarrow{\nabla} H^0(S, \Omega_S^1 \otimes \mathbb{R}^{2r-m-1}\pi_*\Omega_{X/S}^{\bullet < r-1}).]$$

2.2. Nontriviality of transcendental regulators. Now assume to the contrary that $\text{cl}_{2,1}(\xi)(\omega)$ is trivial. Then $\langle \nu, \omega \rangle$ is a period, i.e. $\langle \nu, \omega \rangle = \langle \gamma, \omega \rangle$ for some $\gamma \in H^2(X, \mathbb{Q}(2))$. Applying ∇_u together with the horizontality condition on normal functions, we deduce that

$$(6) \quad \langle \gamma, \nabla_u \omega \rangle = u\langle \gamma, \omega \rangle = u\langle \nu, \omega \rangle = \langle \nabla_u \nu, \omega \rangle + \langle \nu, \nabla_u \omega \rangle = \langle \nu, \nabla_u \omega \rangle.$$

It is well known that the projection of $\nabla_u \omega$ to $H^{1,1}(X_t)$ is the cup product of $\kappa(u)$ and ω , where κ is the Kodaira-Spencer map

$$(7) \quad \Theta_{S,t} \xrightarrow{\kappa} H^1(X_t, \Theta_{X_t})$$

at a point $t \in S$ (Griffiths). The following proposition, which is likely well-known, shows that this cup product is surjective for $K3$ surfaces.

Proposition 2.1. *For a $K3$ surface X , the map*

$$(8) \quad H^1(X, \Theta_X) \otimes H^{2,0}(X) \rightarrow H^{1,1}(X)$$

induced by the contraction $\Theta_X \otimes \wedge^2 \Omega_X^1 \rightarrow \Omega_X^1$ is an isomorphism, where Θ_X is the tangent bundle of X .

Proof. It is instructive to provide a simple proof of this fact. The map (8) gives rise to a pairing

$$(9) \quad H^1(X, \Theta_X) \otimes H^{1,1}(X)^\vee \rightarrow H^{2,0}(X)^\vee.$$

Then (8) is an isomorphism if and only if (9) is a nondegenerate pairing. Combining with Kodaira-Serre duality

$$(10) \quad H^{1,1}(X)^\vee = H^{1,1}(X) \text{ and } H^{2,0}(X)^\vee \simeq H^{0,2}(X),$$

we see that this pairing becomes

$$(11) \quad H^1(X, \Theta_X) \otimes H^1(X, \Omega_X^1) \rightarrow H^2(X, \mathcal{O}_X)$$

which is induced by the nature map $\Theta_X \otimes \Omega_X^1 \rightarrow \mathcal{O}_X$. Therefore, we have the commutative diagram

$$(12) \quad \begin{array}{ccc} H^1(X, \Theta_X) \otimes H^1(X, \Omega_X^1) & \longrightarrow & H^2(X, \mathcal{O}_X) \\ \downarrow \otimes \omega & & \downarrow \otimes \omega \\ H^1(X, \Theta_X) \otimes H^1(X, \Omega_X^1 \otimes K_X) & \longrightarrow & H^2(X, K_X) \end{array}$$

for all $\omega \in H^0(X, K_X)$. The bottom row of (12) is Serre duality and is hence a nondegenerate pairing. Then the nondegeneracy of the top row follows easily when $K_X = \mathcal{O}_X$. \square

Note that $H^1(X, \Theta_X)$ corresponds to all deformations (including non-algebraic) of X . Let $H_{\text{alg}}^1(X, \Theta_X)$ correspond to the algebraic deformations. For a general polarized $K3$ surface (X, L) , $H_{\text{alg}}^1(X, \Theta_X)$ is the subspace $[c_1(L)]^\perp$.

Corollary 2.1. *For a $K3$ surface X , the map*

$$H_{\text{alg}}^1(X, \Theta_X) \otimes H^{2,0}(X) \rightarrow H_v^{1,1}(X),$$

is an isomorphism.

Suppose that the family $\pi : \mathcal{X} \rightarrow S$ is maximum, i.e., the image of the Kodaira-Spencer map κ is $[c_1(L)]^\perp$ at each point $t \in S$. Then by Corollary 2.1, the projections of $\nabla_u \omega$ to $\mathcal{O}_S \otimes R^2 \pi_* \Omega_{\mathcal{X}/S}$, together with ω , generate the subbundle $[c_1(L)]^\perp \cap \mathcal{O}_S \otimes F^1 R^2 \pi_* \mathbb{C}$ as u varies in $H^0(\Theta_S)$. By (6), this cannot happen if the reduced regulator $\underline{\text{cl}}_{2,1}(\xi)$ is non-trivial, which was proven in [C-L1]. Finally, we use the fact that $T_0(S) \simeq H_{\text{alg}}^1(X, \Theta_X)$ together with [C-L1] to deduce the latter statement in Theorem 1.1. Corollary 1.1 follows accordingly.

Proof. (of Theorem 1.2) Let us assume that $\Phi_{r,m}$ is zero. That means that $\text{cl}_{r,m}(\xi)$ is a period with respect to (acting on forms in) $F^{d-r+m+1} H_v^{2d-2r+m+1}(X, \mathbb{C})$. Then from the surjection of

$$H_{\text{alg}}^1(X, \Theta_X) \otimes H_v^{d-r+m+1-\ell, d-r+\ell}(X) \rightarrow H_v^{d-r+m-\ell, d-r+\ell+1}(X),$$

in the case $\ell = 0$, we deduce likewise that $\text{cl}_{r,m}(\xi)$ is a period with respect to $F^{d-r+m} H_v^{2d-2r+m+1}(X, \mathbb{C})$. By iterating the same argument for $\ell = 1, \dots, m-1$, we deduce that $\text{cl}_{r,m}(\xi)$ is a period with respect to $F^{d-r+1} H_v^{2d-2r+m+1}(X, \mathbb{C})$, which implies that $\underline{\text{cl}}_{r,m}(\xi) = 0$. \square

3. DERIVATIVES OF NORMAL FUNCTIONS II

Consider the setting in §1, where $\lambda : \mathcal{X} \rightarrow \mathcal{S}$ is a smooth and proper map of smooth quasi-projective varieties, and where X is a very general member. In this section, we will further assume that \mathcal{S} is affine. Associated to the Gauss-Manin connection ∇ and the algebraic vector fields $H^0(\mathcal{S}, \Theta_{\mathcal{S}})$ is a D -module of differential operators. If $\omega \in H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}} \otimes R^i \lambda_* \mathbb{C}) = H^0(\mathcal{S}, \mathbb{R}^i \lambda_* \Omega_{\mathcal{X}/\mathcal{S}}^\bullet)$ is an algebraic form, one can consider the ideal of partial differential operators with coefficients in $\mathbb{C}(\mathcal{S})$ annihilating ω , which will always be non-zero using the finite dimensionality of cohomology of the fibers of λ and the fact that ∇ is algebraic. This section addresses with the following question.

Question 3.1. *If the transcendental regulator associated to $\Phi_{r,m}(\xi)$ is non-trivial, is the associated normal function ν associated to ξ detectable by a Picard-Fuchs operator $P \in I_\omega$, for some $\omega \in F^{d-r+m+1} H_v^{2d-2r+m+1}(X, \mathbb{C})$; namely is $P\langle \nu, \omega \rangle \neq 0$?*

The answer is a definitive yes in the setting of Theorem 1.2, provided Assumption 3.1 (below) holds. Again, the answer to this question is strongest (unconditional yes) in the case of families of $K3$ surfaces considered in this paper, including product variants such as in Corollary 1.2 and Theorem 1.3. We need the following mild assumption:

Assumption 3.1. *For a fixed choice of r and m above,*

$$\{R_v^{2r-m-1} \lambda_* \mathbb{C}\} \cap \{\mathcal{O}_{\mathcal{S}} \otimes F^r R_v^{2r-m-1} \lambda_* \mathbb{C}\} = 0.$$

As one would expect, this assumption automatically holds in the situation of Theorem 1.1, as well as for the situation of families of $K3$ surfaces in this paper, as well as in Corollary 1.2 and Theorem 1.3.

3.1. Picard-Fuchs equations associated to regulators. Much of the ideas in this section are inspired by [Gr]. Since (again) ∇ is algebraic, everything reduces to a local calculation over a polydisk $S \subset \mathcal{S}$, in the analytic topology. Recall that Θ_S is the holomorphic tangent bundle of S . We can think of Θ_S as the sheaf of holomorphic linear differential operators which naturally carries a ring structure \mathcal{D}_S . That is, it is given by

$$(13) \quad \mathcal{D}_S = \mathcal{O}_S \left[\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \dots, \frac{\partial}{\partial z_n} \right]$$

where $\mathcal{O}_S = \mathbb{C}[[z_1, z_2, \dots, z_n]]$. By identifying $\partial/\partial z_k$ with $\nabla_{\partial/\partial z_k}$, \mathcal{D}_S acts on $\mathcal{O}_S \otimes R^q \pi_* \mathbb{C}$ via

$$(14) \quad (v_1 v_2 \dots v_l) \omega = \nabla_{v_1} \nabla_{v_2} \dots \nabla_{v_l} \omega$$

for $v_1, v_2, \dots, v_l \in H^0(\Theta_S)$, where we write $H^0(-)$ for $H^0(S, -)$. For $\omega \in H^0(\mathcal{O}_S \otimes R^q \pi_* \mathbb{C})$, we let I_ω be the *Picard-Fuchs ideal* annihilating ω , i.e., the left-side ideal consisting of differential operators $P \in H^0(\mathcal{D}_S)$ satisfying $P\omega = 0$.

As in §2 let us again for simplicity restrict to the situation of a family of $K3$ surfaces. We fix a nonzero section $\omega \in H^0(K_{\mathcal{X}/\mathcal{S}})$. For all $u \in H^0(\Theta_S)$ and all Picard-Fuchs operators $P \in H^0(\mathcal{D}_S)$ such that $P(\nabla_u \omega) = 0$, i.e., $P \in I_{\nabla_u \omega}$, it is obvious that

$$(15) \quad (Pu)\omega = 0$$

and hence

$$(16) \quad (Pu)\langle\gamma, \omega\rangle = 0$$

for all $\gamma \in H^2(X, \mathbb{C})$ (where $H^2(X, \mathbb{C})$ is identified with $H^0(S, R^2\pi_*\mathbb{C})$, using S a polydisk). Again let $\xi \in \text{CH}^2(\mathcal{X}/S, 1)$ be the result of an algebraic deformation of a cycle in the central fiber X restricted to \mathcal{X}/S , and $\text{cl}_{2,1}$ be the regulator map

$$(17) \quad \text{cl}_{2,1} : \text{CH}^2(\mathcal{X}/S, 1) \rightarrow H^0\left(\frac{\mathcal{O}_S \otimes R^2\pi_*\mathbb{C}}{\mathcal{O}_S \otimes F^2R^2\pi_*\mathbb{C} + R^2\pi_*\mathbb{Q}(2)}\right).$$

We let ν be a lift of $\text{cl}_{2,1}(\xi)$ to $H^0(\mathcal{O}_S \otimes R^2\pi_*\mathbb{C})$.

For $P \in I_{\nabla_u\omega}$, $Pu \in I_\omega$ “kills” all the periods $\langle\gamma, \omega\rangle$ for $\gamma \in H^2(X, \mathbb{Q}(2))$. Therefore, $(Pu)\langle\nu, \omega\rangle$ is independent of the choice of the lifting of $\text{cl}_{2,1}(\xi)$. Obviously, we have

$$(18) \quad Pu\langle\nu, \omega\rangle = P(\langle\nabla_u\nu, \omega\rangle + \langle\nu, \nabla_u\omega\rangle).$$

Since $\langle\nabla_u\nu, \omega\rangle = 0$, (18) becomes

$$(19) \quad P(u\langle\nu, \omega\rangle - \langle\nu, \nabla_u\omega\rangle) = 0$$

which is a system of differential equations satisfied by $\text{cl}_{2,1}(\xi)$. We put this into the following proposition.

Proposition 3.1. *Let \mathcal{X}/S , ν and ω be given as above. Then (19) holds for all $u \in H^0(\Theta_S)$ and $P \in I_{\nabla_u\omega}$. Or equivalently,*

$$(20) \quad u\langle\nu, \omega\rangle - \langle\nu, \nabla_u\omega\rangle = \langle\gamma, \nabla_u\omega\rangle$$

for some $\gamma \in H^2(X, \mathbb{C}) = H^0(S, R^2\pi_*\mathbb{C})$.

Here we need to say something about (20). Namely, we want to say that the solutions of $P\gamma = 0$ for $P \in I_{\nabla_u\omega}$ are generated by $\langle\gamma, \nabla_u\omega\rangle$ for all $\nabla\gamma = 0$. Roughly, it follows from [Gr](1.28). It is actually more elementary than that as a consequence of the following observation, which is a generalization of the fact that a function with vanishing derivative is constant.

Lemma 3.1. *Let E be a flat holomorphic vector bundle over the polydisk S with flat connection ∇ and let I_η be the Picard-Fuchs ideal associated to an $\eta \in H^0(E)$ defined as above. Then the solutions of the system of differential equations $P\gamma = 0$ for $P \in I_\eta$ are generated as a vector space over \mathbb{C} by $\langle\gamma, \eta\rangle$ for all $\gamma \in E^\vee$ with $\nabla\gamma = 0$, where E^\vee is the dual of E .*

3.2. Nontriviality of Picard-Fuchs operators. Suppose that any Picard-Fuchs operator in I_ω annihilates $\text{cl}_{2,1}(\xi)(\omega)$. Then $\langle\nu, \omega\rangle = \langle\gamma, \omega\rangle$ for some $\gamma \in H^2(X, \mathbb{C})$. It follows that

$$(21) \quad (Pu)\langle\nu, \omega\rangle = 0$$

for all $u \in H^0(\Theta_S)$ and $P \in I_{\nabla_u\omega}$. By Proposition 3.1, we have

$$(22) \quad (Pu)\langle\nu, \omega\rangle = P\langle\nu, \nabla_u\omega\rangle = 0$$

for all $u \in H^0(\Theta_S)$ and $P \in I_{\nabla_u\omega}$ and

$$(23) \quad \langle\nu, \nabla_u\omega\rangle = \langle\gamma, \nabla_u\omega\rangle$$

for some $\gamma \in H^2(X, \mathbb{C})$. Equivalently, on $H^0(\mathcal{O}_S \otimes R^2\pi_*\mathbb{C})$ and again after identifying $H^2(X, \mathbb{C})$ with $H^0(S, R^2\pi_*\mathbb{C})$ (recall again S is a polydisk), we have

$$(24) \quad \nu \in [\nabla_u \omega]^\perp + H^2(X, \mathbb{C}).$$

That is, $\text{cl}_{2,1}(\xi)$ lies in the image of $[\nabla_u \omega]^\perp + H^2(X, \mathbb{C})$, which we simply write as

$$(25) \quad \text{cl}_{2,1}(\xi) \in [\nabla_u \omega]^\perp + H^2(X, \mathbb{C}).$$

Assume for the moment that the family $\pi : \mathcal{X} \rightarrow S$ is *maximal*, i.e., the image of the Kodaira-Spencer map κ is $[c_1(L)]^\perp$ at each point $t \in S$. Then by Corollary 2.1, the projections of $\nabla_u \omega$ to $\mathcal{O}_S \otimes R^1\pi_*\Omega_{\mathcal{X}/S}$ generate the subbundle $[c_1(L)]^\perp$ as u varies in $H^0(\Theta_S)$. And since

$$(26) \quad \text{cl}_{2,1}(\xi) \in \bigcap_{u \in H^0(\Theta_S)} ([\nabla_u \omega]^\perp + H^2(X, \mathbb{C})) \cap ([\omega]^\perp + H^2(X, \mathbb{C}))$$

and $c_1(L) \in H^2(X, \mathbb{C})$, we see that

$$(27) \quad \text{cl}_{2,1}(\xi) \in H^2(X, \mathbb{C}),$$

i.e. the normal function has zero infinitesimal invariant, and hence zero topological invariant by [Sa]. Let us explain this more precisely. If we consider for the moment the general setting in §1 of a smooth and proper morphism $\lambda : \mathcal{X} \rightarrow \mathcal{S}$ of smooth quasi-projective varieties, where say \mathcal{S} is affine, with space of normal functions $\text{Ext}_{V\text{MHS}}^1(\mathbb{Q}(0), R^{2r-m-1}\lambda_*\mathbb{Q}(r))$, then there is a short exact sequence:

$$0 \rightarrow J(H_f^{2r-m-1}(X, \mathbb{Q}(r))) \rightarrow \text{Ext}_{V\text{MHS}}^1(\mathbb{Q}(0), R^{2r-m-1}\lambda_*\mathbb{Q}(r)) \xrightarrow{\delta} \text{hom}_{\text{MHS}}(\mathbb{Q}(0), H^1(\mathcal{S}, R^{2r-m-1}\lambda_*\mathbb{Q}(r))) \rightarrow 0,$$

which induces an injection

$$\text{Ext}_{V\text{MHS}}^1(\mathbb{Q}(0), R_v^{2r-m-1}\lambda_*\mathbb{Q}(r)) \hookrightarrow \text{hom}_{\text{MHS}}(\mathbb{Q}(0), H^1(\mathcal{S}, R_v^{2r-m-1}\lambda_*\mathbb{Q}(r))),$$

together with an injection (using \mathcal{S} affine, see [Sa])

$$\text{hom}_{\text{MHS}}(\mathbb{Q}(0), H^1(\mathcal{S}, R_v^{2r-m-1}\lambda_*\mathbb{Q}(r))) \hookrightarrow \nabla \Gamma J,$$

where

$$\nabla \Gamma J := \frac{\ker \nabla : H^0(\mathcal{S}, \Omega_S^1 \otimes F^{r-1}R_v^{2r-m-1}\lambda_*\mathbb{C}) \rightarrow H^0(\mathcal{S}, \Omega_S^2 \otimes F^{r-2}R_v^{2r-m-1}\lambda_*\mathbb{C})}{\nabla H^0(\mathcal{S}, \mathcal{O}_S \otimes F^r R_v^{2r-m-1}\lambda_*\mathbb{C})}.$$

For $\nu \in \text{Ext}_{V\text{MHS}}^1(\mathbb{Q}(0), R^{2r-m-1}\lambda_*\mathbb{Q}(r))$, $\delta\nu$ gives the topological invariant of ν . Next, consider the sheaf

$$\nabla J := \frac{\ker \nabla : \Omega_S^1 \otimes F^{r-1}R_v^{2r-m-1}\lambda_*\mathbb{C} \rightarrow \Omega_S^2 \otimes F^{r-2}R_v^{2r-m-1}\lambda_*\mathbb{C}}{\nabla(\mathcal{O}_S \otimes F^r R_v^{2r-m-1}\lambda_*\mathbb{C})},$$

with corresponding $\Gamma \nabla J := H^0(\mathcal{S}, \nabla J)$. By definition of a normal function, one has the Griffiths infinitesimal invariant $\delta_G \nu \in \Gamma \nabla J$. Under Assumption 3.1, the natural map $\nabla \Gamma J \rightarrow \Gamma \nabla J$ is an isomorphism. Indeed, by Assumption 3.1, this follows from the short exact sequence:

$$0 \rightarrow \mathcal{O}_S \otimes F^r R_v^{2r-m-1}\lambda_*\mathbb{C} \xrightarrow{\nabla} (\Omega_S^1 \otimes F^{r-1}R_v^{2r-m-1}\lambda_*\mathbb{C})_{\ker \nabla} \rightarrow \nabla J \rightarrow 0.$$

Now back to the case of our family of $K3$ surfaces, with $(d, r, m) = (2, 2, 1)$, Assumption 3.1 automatically holds, and the normal function ν associated to $\text{cl}_{2,1}(\xi)$, satisfies $\delta_G \nu = 0$, hence $\delta\nu = 0$. This implies that the normal function arises from the fixed part $J(H_f^2(X, \mathbb{Q}(2)))$. This cannot happen since the reduced regulator

$\underline{\text{cl}}_{2,1}(\xi)$ is non-trivial. Finally, we use the fact that $T_0(S) \simeq H_{\text{alg}}^1(X, \Theta_X)$ together with [C-L1] to deduce the non-triviality of the Picard-Fuchs operator acting on a normal function arising from the general subfamilies in the latter statement in Theorem 1.1. A similar story holds in the setting of Corollary 1.1, (and as will be clearer later, as well as in Corollary 1.2 and Theorem 1.3). Quite generally, in the setting of Theorem 1.2, let us assume that any Picard-Fuchs operator applied to $\Phi_{r,m}$ is zero. Then from the surjection of

$$H_{\text{alg}}^1(X, \Theta_X) \otimes H_v^{d-r+m+1-\ell, d-r+\ell}(X) \rightarrow H_v^{d-r+m-\ell, d-r+\ell+1}(X),$$

in the case $\ell = 0$, we deduce as in (26) that

$$\text{cl}_{r,m}(\xi) \in [H^0(\mathcal{O}_S \otimes F^{d-r+m} R^{2d-2r+m+1} \pi_* \mathbb{C})]^\perp + H^{2d-2r+m+1}(X, \mathbb{C}).$$

By iterating the same argument for $\ell = 1, \dots, m-1$, we deduce that

$$\text{cl}_{r,m}(\xi) \in [H^0(\mathcal{O}_S \otimes F^{d-r+1} R^{2d-2r+m+1} \pi_* \mathbb{C})]^\perp + H^{2d-2r+m+1}(X, \mathbb{C}),$$

which implies that the associated normal function has zero infinitesimal invariant, and thus $\underline{\text{cl}}_{r,m}(\xi) = 0$, which is not the case.

4. PROOF OF THEOREM 1.3

In this section we restrict to the case where X is a projective $K3$ surface. We recall the real regulator

$$(28) \quad r_{3,1} : \text{CH}^3(X \times X, 1) \rightarrow H^{2,2}(X \times X, \mathbb{R}(2)).$$

The image of $r_{3,1}$ thus contains

$$(29) \quad r_{3,1}(\text{CH}^1(X) \otimes \text{CH}^2(X, 1)) \otimes \mathbb{R} = H^{1,1}(X, \mathbb{Q}(1)) \otimes H^{1,1}(X, \mathbb{R}(1))$$

for X general and it also contains the class $[\Delta_X]$ of the diagonal. So it is natural to look at the reduced real regulator

$$(30) \quad \underline{r}_{3,1} : \text{CH}^3(X \times X, 1) \xrightarrow{r_{3,1}} H^{2,2}(X \times X, \mathbb{R}) \xrightarrow{\text{projection}} V_X$$

where

$$(31) \quad V_X = H^{2,2}(X \times X, \mathbb{R}) \cap (H^{1,1}(X, \mathbb{Q}(1)) \otimes H^{1,1}(X, \mathbb{R}(1)))^\perp \\ \cap (H^{1,1}(X, \mathbb{R}(1)) \otimes H^{1,1}(X, \mathbb{Q}(1)))^\perp \cap [\Delta_X]^\perp.$$

It was proven in [C-L2] that

$$(32) \quad \text{Im}(\underline{r}_{3,1}) \otimes \mathbb{R} \neq 0.$$

Of course, this implies that the indecomposables

$$(33) \quad \text{CH}_{\text{ind}}^3(X \times X, 1) \otimes \mathbb{Q} \neq 0$$

for a general projective $K3$ surface X [C-L2, Corollary 1.3].

Now let us look at the transcendental part of $\text{cl}_{3,1}$:

$$(34) \quad \Phi_{3,1} : \text{CH}^3(X \times X, 1) \rightarrow \frac{\{F^3(H_v^2(X, \mathbb{C}) \otimes H_v^2(X, \mathbb{C}))\}^\vee}{H_4(X \times X, \mathbb{Q}(1))},$$

where now X is a very general $K3$ and $H_v^2(X, \mathbb{C})$ is transcendental cohomology. Although one can follow the same argument in [C-L2] to prove that $\Phi_{3,1}$ is non-trivial by a degeneration argument, there is an easier way to derive this from Theorem 1.2.

The proof of Corollary 1.2 is a stepping stone to the proof of the stronger Theorem 1.3.

4.1. Non-triviality of $\Phi_{3,1}$. It is instructive to explain precisely how Theorem 1.2 leads Corollary 1.2, viz., to the non-triviality of $\Phi_{3,1}$ for $Y := X \times X$, where X is a very general projective $K3$ surface. In this case Y takes the role of X in Theorem 1.1, with $(d, r, m, \ell) = (4, 3, 1, 0)$, $H_{\text{alg}}^1(Y, \Theta_Y)$ will be identified with $H_{\text{alg}}^1(X, \Theta_X) \simeq \mathbb{C}^{19}$, and $H_v^{2d-2r+m+1}(Y, \mathbb{Q}) = H_v^4(Y, \mathbb{Q})$ will be replaced by

$$[\Delta_X]^\perp \cap \{H_v^2(X, \mathbb{Q}) \otimes H_v^2(X, \mathbb{Q})\},$$

where $[\Delta_X]$ is the diagonal class. The pairing in Theorem 1.2 amounts to studying the properties of the pairing

$$H^1(\Theta_X) \otimes H^{3,1}(X \times X) \rightarrow H^{2,2}(X \times X),$$

which amounts to a Gauss-Manin derivative calculation. So let \mathcal{X}/S be a smooth projective family of $K3$ surfaces over a polydisk S (arising from a universal family), $\mathcal{Y} = \mathcal{X} \times_S \mathcal{X}$, $X = \mathcal{X}_0$ be a very general fiber of \mathcal{X}/S , $Y = X \times X$ and $\pi_{\mathcal{X}}$ be the projection $\mathcal{X} \rightarrow S$. Let ∇ be the GM connection associated to \mathcal{X}/S and let $\alpha \in H^1(\Theta_X)$ be a tangent vector of S at 0. For $\omega \in H^0((\pi_{\mathcal{X}})_* \wedge^2 \Omega_{\mathcal{X}/S})$ and $\eta \in H^0(R^1(\pi_{\mathcal{X}})_* \Omega_{\mathcal{X}/S})$, i.e., for $\omega \in H^{2,0}(X)$ and $\eta \in H^{1,1}(X)$ when restricted to X , we claim that

$$(35) \quad \bigcap_{\alpha, \omega, \eta} \left((\nabla_\alpha(\omega \otimes \eta))^\perp \cap (\nabla_\alpha(\eta \otimes \omega))^\perp \right) \cap [\Delta_X]^\perp = \{0\}$$

in $H^{2,2}(Y)$ and hence the condition on the cup product pairing in Theorem 1.2 holds. Note that

$$(36) \quad [\nabla_\alpha(\omega \otimes \eta)] = [\nabla_\alpha \omega] \otimes \eta + \omega \otimes [\nabla_\alpha \eta]$$

where $[\nabla_\alpha(\omega \otimes \eta)]$, $[\nabla_\alpha \omega]$ and $[\nabla_\alpha \eta]$ are the projections of $\nabla_\alpha(\omega \otimes \eta)$, $\nabla_\alpha \omega$ and $\nabla_\alpha \eta$ onto $H^{2,2}(Y)$, $H^{1,1}(X)$ and $H^{0,2}(X)$, respectively. We know that

$$(37) \quad [\nabla_\alpha \omega] = \langle \alpha, \omega \rangle \text{ and } [\nabla_\alpha \eta] = \langle \alpha, \eta \rangle$$

where $\langle \bullet, \bullet \rangle$ is the pairing

$$(38) \quad H^1(\Theta_X) \otimes (H^{1,1}(X) \oplus H^{2,0}(X)) \rightarrow H^{0,2}(X) \oplus H^{1,1}(X).$$

We write $\langle \alpha, \omega \rangle = \delta_\alpha \omega$ and $\langle \alpha, \eta \rangle = \delta_\alpha \eta$. Then (35) follows directly from the following statement.

Proposition 4.1. *For every complex $K3$ surface X ,*

$$(39) \quad \bigcap_{\alpha, \omega, \eta} \left((\delta_\alpha \omega \otimes \eta + \omega \otimes \delta_\alpha \eta)^\perp \cap (\delta_\alpha \eta \otimes \omega + \eta \otimes \delta_\alpha \omega)^\perp \right) \cap [\Delta_X]^\perp = \{0\}$$

in $H^{2,2}(X \times X, \mathbb{C})$, where $\alpha \in H^1(\Theta_X)$, $\omega \in H^{2,0}(X)$ and $\eta \in H^{1,1}(X)$.

Proof. Combining Proposition 2.1 with the fact that

$$(40) \quad \langle \delta_\alpha \omega, \eta \rangle + \langle \omega, \delta_\alpha \eta \rangle = 0,$$

we obtain

$$(41) \quad \langle [\Delta_X], \delta_\alpha \omega \otimes \eta + \omega \otimes \delta_\alpha \eta \rangle = 0$$

and hence

$$(42) \quad \begin{aligned} & \text{Span}\{\delta_\alpha \omega \otimes \eta + \omega \otimes \delta_\alpha \eta\} \\ &= [\Delta_X]^\perp \cap (H^{1,1}(X) \otimes H^{1,1}(X) \oplus H^{2,0}(X) \otimes H^{0,2}(X)). \end{aligned}$$

Similarly,

$$(43) \quad \begin{aligned} & \text{Span}\{\delta_\alpha \eta \otimes \omega + \eta \otimes \delta_\alpha \omega\} \\ &= [\Delta_X]^\perp \cap (H^{0,2}(X) \otimes H^{2,0}(X) \oplus H^{1,1}(X) \otimes H^{1,1}(X)) \end{aligned}$$

and (39) follows easily. \square

Note that $H_f^2(X, \mathbb{C}) = H^{1,1}(X, \mathbb{Q}(1)) \otimes \mathbb{C}$ and $\pi_1(S)$ acts on $H_v^2(X, \mathbb{C})$ irreducibly. It is then not hard to see that

$$(44) \quad H_f^4(Y, \mathbb{C}) \cap H_v^2(X, \mathbb{C}) \otimes H_v^2(X, \mathbb{C}) \cap [\Delta_X]^\perp = \{0\}$$

and hence

$$(45) \quad H_f^4(Y, \mathbb{C}) \subset V_X^\perp.$$

Since $r_{3,1}(\xi) \neq 0$, this shows that $\Phi_{3,1}$ is non-trivial.

4.2. The purely transcendental regulator $\Psi_{3,1}$. We now turn our attention to the proof of Theorem 1.3. More explicitly, we fix a nonvanishing holomorphic 2-form $\omega \in H^{2,0}(X)$ and look at

$$(46) \quad \langle \text{cl}_{3,1}(\xi), \omega \otimes \omega \rangle$$

modulo the periods $\int_\gamma \omega \otimes \omega$ for $\gamma \in H_4(X \times X, \mathbb{Q}(1))$. We claim $\Psi_{3,1}$ is non-trivial, or equivalently, $\langle \text{cl}_{3,1}(\xi), \omega \otimes \omega \rangle$ is not a period for some $\xi \in \text{CH}^3(X \times X, 1)$. Here we go slightly beyond the range of ℓ in Theorem 1.2, namely we allow $\ell = -1, 0$. More specifically we consider

$$(47) \quad \begin{aligned} H_{\text{alg}}^1(Y, \Theta_Y) &\rightarrow \text{hom}(H^{4,0}(Y), H^{3,1}(Y)), \\ H_{\text{alg}}^1(Y, \Theta_Y)^{\otimes 2} &\rightarrow \text{hom}(H^{4,0}(Y), H^{2,2}(Y)), \end{aligned}$$

where again $Y = X \times X$ is a self product of a very general projective $K3$ surface X , and $H_{\text{alg}}^1(Y, \Theta_Y)$ is identified with the first order deformation space of a universal family of projective $K3$'s. Of course if the former map in (47) were surjective, then the latter map could be replaced by

$$H_{\text{alg}}^1(Y, \Theta_Y) \rightarrow \text{hom}(H^{3,1}(Y), H^{2,2}(Y)).$$

Let us assume for the moment that both maps in (47) are surjective. Then by the same reasoning as in the previous section, one could argue that $\Psi_{3,1}$ is non-trivial. However by a dimension count, it is clear that both maps in (47) are not surjective. We remedy this by passing to the symmetric product $\hat{Y} = Y/\langle \sigma \rangle$, where $\langle \sigma \rangle$ is the symmetric group of order 2 acting on $Y = X \times X$. In fact, instead of working directly on \hat{Y} , we will work with the equivariant cohomologies $H^4(Y, \mathbb{Q})^\sigma$, and $\text{CH}^3(Y, 1)^\sigma$. That is, they consist of classes fixed under σ . Note that $H^4(Y, \mathbb{Q})^\sigma$ is still a Hodge structure. With the same setup for $\Phi_{3,1}$ and following the same argument by differentiating, we consider the orthogonal complements

$$(\nabla_\alpha(\omega \otimes \omega))^\perp \text{ and } (\nabla_\beta \nabla_\alpha(\omega \otimes \omega))^\perp,$$

following the situation in (47). In particular, we are interested in the subspace

$$(48) \quad \bigcap_{\alpha, \beta} (\delta_\alpha \delta_\beta \omega \otimes \omega + \delta_\alpha \omega \otimes \delta_\beta \omega + \delta_\beta \omega \otimes \delta_\alpha \omega + \omega \otimes \delta_\alpha \delta_\beta \omega)^\perp \cap \bigcap_{\alpha} (\delta_\alpha \omega \otimes \omega + \omega \otimes \delta_\alpha \omega)^\perp \cap (\omega \otimes \omega)^\perp \cap [\Delta_X]^\perp$$

when restricted to Y . Note that

$$(49) \quad \langle \delta_\alpha \omega, \delta_\beta \omega \rangle + \langle \omega, \delta_\alpha \delta_\beta \omega \rangle = \langle \delta_\alpha \omega, \delta_\beta \omega \rangle + \langle \omega, \delta_\beta \delta_\alpha \omega \rangle = 0$$

by (40) and hence

$$(50) \quad \delta_\alpha \delta_\beta \omega \otimes \omega + \delta_\alpha \omega \otimes \delta_\beta \omega + \delta_\beta \omega \otimes \delta_\alpha \omega + \omega \otimes \delta_\alpha \delta_\beta \omega \in [\Delta_X]^\perp$$

for all $\alpha, \beta \in H^1(\Theta_X)$. Similarly,

$$(51) \quad \delta_\alpha \omega \otimes \omega + \omega \otimes \delta_\alpha \omega \in [\Delta_X]^\perp$$

for all $\alpha \in H^1(\Theta_X)$. Although we do not need it, (49) also implies that $\delta_\alpha \delta_\beta = \delta_\beta \delta_\alpha$ and hence the map

$$(52) \quad H^1(\Theta_X) \otimes H^1(\Theta_X) \rightarrow \text{hom}(H^{2,0}(X), H^{0,2}(X))$$

induced by $H^1(\Theta_X) \otimes H^1(\Theta_X) \otimes H^{2,0}(X) \rightarrow H^{0,2}(X)$ is a symmetric nondegenerate pairing. Obviously,

$$(53) \quad \begin{aligned} & \text{Span}\{\delta_\alpha \delta_\beta \omega \otimes \omega + \delta_\alpha \omega \otimes \delta_\beta \omega + \delta_\beta \omega \otimes \delta_\alpha \omega + \omega \otimes \delta_\alpha \delta_\beta \omega\} \\ &= [\Delta_X]^\perp \cap H^{2,2}(Y)^\sigma \end{aligned}$$

and

$$(54) \quad \text{Span}\{\delta_\alpha \omega \otimes \omega + \omega \otimes \delta_\alpha \omega\} = [\Delta_X]^\perp \cap H^{3,1}(Y)^\sigma$$

by (50), (51) and the nondegeneracy of (52). Therefore,

$$(55) \quad \begin{aligned} & \bigcap_{\alpha, \beta} (\delta_\alpha \delta_\beta \omega \otimes \omega + \delta_\alpha \omega \otimes \delta_\beta \omega + \delta_\beta \omega \otimes \delta_\alpha \omega + \omega \otimes \delta_\alpha \delta_\beta \omega)^\perp \cap \\ & \bigcap_{\alpha} (\delta_\alpha \omega \otimes \omega + \omega \otimes \delta_\alpha \omega)^\perp \cap (\omega \otimes \omega)^\perp \cap [\Delta_X]^\perp \cap H^4(Y, \mathbb{C})^\sigma = \{0\}. \end{aligned}$$

Thus, in order to prove Theorem 1.3, we just have to find ξ such that $\mathfrak{L}_{3,1}(\xi) \neq 0$ and $\text{cl}_{3,1}(\xi) \in H^4(Y, \mathbb{C})^\sigma$. The obvious way to do this is to find an equivariant higher Chow class $\xi \in \text{CH}^3(Y, 1)^\sigma$ with $\mathfrak{L}_{3,1}(\xi) \neq 0$. Namely, we need a slightly stronger statement than (32). That is,

Theorem 4.1. *There exists $\xi \in \text{CH}^3(X \times X, 1)^\sigma$ such that $\mathfrak{L}_{3,1}(\xi) \neq 0$ for a general projective K3 surface X .*

Proof. This is a consequence of the explicit construction of the cycle in [C-L2]. \square

5. INTERMEZZO: LATTICE POLARIZED K3 SURFACES, HYPERSURFACE NORMAL FORMS, AND MODULAR PARAMETRIZATION

At this point it is natural to ask how one might construct explicit families of K3 surfaces satisfying the conditions of Theorem 1.1, with enough “internal structure” to make it possible to construct explicit cycles with nontrivial $\Phi_{2,1}$. In light of §3, it would also be highly desirable to have a means of explicitly constructing the Picard-Fuchs operators for these families.

Families of the sort required by Theorem 1.1 with a fixed generic Néron-Severi lattice are known as *lattice polarized K3 surfaces* [Dol]. Let X be an algebraic K3 surface over the field of complex numbers. If M is an even lattice of signature $(1, \ell-1)$ (with $\ell > 0$), then an M -polarization on X is a primitive lattice embedding

$$i : M \hookrightarrow \text{NS}(X)$$

such that the image $i(M)$ contains a pseudo-ample class. There is also a coarse moduli space \mathcal{M}_M for equivalence classes of pairs (M, i) , which satisfies a version of the global Torelli theorem. Moreover, surjectivity of the period map holds for families which are maximum in the sense of §3.2; any family whose image in \mathcal{M}_M is surjective satisfies this condition.

An elliptic K3 surface with section consists of a triple (X, ϕ, S) of a K3 surface X , an elliptic fibration $\phi : X \rightarrow \mathbb{P}^1$, and a smooth rational curve $S \subset X$ forming a section of ϕ . This “internal structure” of an elliptic fibration with section on a K3 surface X is equivalent to a lattice polarization of X by the even rank two hyperbolic lattice

$$H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(see [C-D1, Theorem 2.3] for details). The moduli space \mathcal{M}_H of H -polarized K3 surfaces has complex dimension 18, and the generic elliptic K3 surface with section has 24 singular fibers of Kodaira type I_1 . Instead of working with a very general member of this family, which will have Picard rank $\ell = 2$, one can enhance the lattice polarization by considering a higher rank lattice M , with H as a sublattice. For each distinct embedding of H into M , up to automorphisms of the ambient lattice M , we find an elliptic surface structure with section on all M -polarized K3 surfaces. There is a decomposition of the Néron-Severi lattice

$$\text{NS}(X) = H \oplus W_X ,$$

where W_X is the negative definite sublattice of $\text{NS}(X)$ generated by classes associated to algebraic cycles orthogonal to both the elliptic fiber and the section. The sublattice

$$W_X^{\text{root}} := \{r \in W_X \mid \langle r, r \rangle = -2\}$$

is called the *ADE type* of the elliptic fibration with section, as it decomposes naturally into the sum of ADE type sublattices spanned by c_1 of the irreducible (rational) components of the singular fibers of the elliptic fibration (see [C-D1, Section 6]).

For the explicit computations in §6 and §7 we will make essential use of one particular elliptic fibration with section on a family of K3 surfaces polarized by the lattice $H \oplus E_8 \oplus E_8$. It is not, in fact, the “standard” fibration, which corresponds to $W_X = E_8 \oplus E_8$, but the “alternate fibration” for which $W_X = D_{16}^+$ (the other even negative definite rank 16 lattice). Up to ambient lattice automorphisms, these are the only two distinct embeddings of the lattice H into $H \oplus E_8 \oplus E_8$. As a result, we know that these are the only two elliptic fibrations with section on a very general member of this family of K3 surfaces [C-D2].

5.1. Normal forms and elliptic fibrations. The natural setting for Theorem 1.1 is families of lattice-polarized K3 surfaces which cover their corresponding coarse moduli spaces. In order to effectively compute, we first need to construct such maximal families of K3 surfaces.

The most classical construction of $K3$ surfaces is as smooth quartic (anticanonical) hypersurfaces in \mathbb{P}^3 . A very general member of this family will have a 4-polarization and Picard rank $\ell = 1$. It is possible, however, to construct subfamilies of smooth quartics with natural polarization by lattices of much higher rank. For example, consider the “Fermat quartic pencil”

$$(56) \quad X_t := \{x^4 + y^4 + z^4 + w^4 + t \cdot xyzw = 0\} \subset \mathbb{P}^3.$$

For generic $t \in \mathbb{P}^1$, the group $G := (\mathbb{Z}/4\mathbb{Z})^2$ acts on X_t by

$$x \mapsto \lambda \cdot x, \quad y \mapsto \mu \cdot y, \quad z \mapsto \lambda^{-1} \mu^{-1} \cdot z,$$

where λ and μ are fourth roots of unity.

The induced action of this group on the cohomology of X_t fixes the holomorphic two-form ω_t (i.e., it acts symplectically). Nikulin’s classification of symplectic actions on $K3$ surfaces then implies that there is a rank 18 negative definite sublattice in the Néron-Severi group of X_t , which together with the (fixed) 4-polarization class means that the Picard rank of X_t is at least 19. As the family is not isotrivial, the Picard rank is not generically equal to 20, and we conclude that the family $X_t, t \in \mathbb{P}^1$ satisfies the conditions of Theorem 1.1 with $\ell = 19$. (See [Wh] for a general set of tools to bound the Picard rank of pencils of hypersurfaces with a high degree of symmetry.) This is an example of a *normal form* for the corresponding class of lattice polarized $K3$ surfaces, in this case providing a natural generalization of the Hesse pencil normal form for cubic curves in \mathbb{P}^2 .

There is another family Y_t of $K3$ surfaces with $\ell = 19$ easily derivable from the X_t in (56) by quotienting each X_t by the group G and simultaneously resolving the resulting singularities in the family. The family Y_t , known as the “quartic mirror family,” has rank 19 lattice polarization by the lattice $M_2 := H \oplus E_8 \oplus E_8 \oplus \langle -4 \rangle$.

Another way to construct families of 4-polarized $K3$ surfaces with an enhanced lattice polarization is to consider *singular* quartic hypersurfaces in \mathbb{P}^3 . By introducing ordinary double point singularities of ADE type, it is a simple matter to engineer (upon minimal resolution) $K3$ surfaces with large negative definite sublattices of ADE type in their Néron-Severi groups. One feature that both the smooth and singular quartic hypersurface constructions enjoy is that for each line lying on the surface there is a corresponding elliptic fibration structure, defined by taking the pencil of planes passing through the line and considering the excess intersection of each (a pencil of cubic curves). In this way, suitably nice quartic normal forms readily admit the structure of elliptic fibrations with section corresponding to various embeddings of the hyperbolic lattice H into their polarizing lattices.

Let us illustrate this with the key example for the constructions in §6 and §7, the singular quartic normal form for $K3$ surfaces polarized by the lattice

$$M := H \oplus E_8 \oplus E_8$$

[C-D2]. Let (X, i) be an M -polarized $K3$ surface. Then there exists a triple $(a, b, d) \in \mathbb{C}^3$, with $d \neq 0$ such that (X, i) is isomorphic to the minimal resolution of the quartic surface

$$Q_M(a, b, d) : y^2zw - 4x^3z + 3axzw^2 + b zw^3 - \frac{1}{2}(dz^2w^2 + w^4) = 0.$$

Two such quartics $Q_M(a_1, b_1, d_1)$ and $Q_M(a_2, b_2, d_2)$ determine via minimal resolution isomorphic M -polarized $K3$ surfaces if and only if

$$(a_2, b_2, d_2) = (\lambda^2 a_1, \lambda^3 b_1, \lambda^6 d_1)$$

for some $\lambda \in \mathbb{C}^*$. Thus the coarse moduli space for M -polarized $K3$ surfaces is the open variety

$$\mathcal{M}_M = \{[a, b, d] \in \mathbb{WP}(2, 3, 6) \mid d \neq 0\}$$

with fundamental invariants

$$\frac{a^3}{d} \text{ and } \frac{b^2}{d} .$$

On the singular quartic hypersurface $Q_M(a, b, d) \subset \mathbb{P}^3$ there are two distinct lines

$$\{x = w = 0\} \text{ and } \{z = w = 0\} ,$$

and the points

$$P_1 := [0, 1, 0, 0] \text{ and } P_2 := [0, 0, 1, 0]$$

are rational double point singularities on $Q_M(a, b, d)$ of ADE types A_{11} and E_6 respectively. The standard fibration is induced by the projection to $[z, w]$, and the alternate fibration is induced by the projection to $[x, w]$. Moreover, among the exceptional rational curves in the resolution of P_1 are sections of both elliptic fibrations on $X(a, b, d)$; among the exceptional rational curves in the resolution of P_2 is a second section of the alternate fibration on $X(a, b, d)$.

It is useful to note that both the quartic mirror normal form Y_t for M_2 -polarized $K3$ surfaces and the M -polarized normal form $X(a, b, d)$ admit natural reinterpretations as the generic anticanonical hypersurfaces in certain toric Fano varieties [Dor1, Dor2, CDLW]. In both cases we build the toric Fano variety from the normal fan of a reflexive polytope. For the M_2 -polarized case, the polytope is the convex hull of

$$\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -1, -1)\} \subset \mathbb{R}^3 ,$$

polar to the Newton polytope for \mathbb{P}^3 . For the M -polarized case, the polytope is the convex hull of

$$\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -4, -6)\} ,$$

polar to the Newton polytope for $\mathbb{WP}(1, 1, 4, 6)$. What is more, the two elliptic fibrations with section on a very general $X(a, b, d)$ are themselves induced by ambient toric fibrations on the toric variety in which it sits as a hypersurface. Combinatorially, these correspond to reflexive ‘‘slices’’ of the corresponding polytope, i.e., planes in \mathbb{R}^3 which slice the reflexive polytope in a reflexive polygon.

5.2. Picard-Fuchs equations and modular parametrization. There is a reverse nesting of moduli spaces corresponding to embeddings of the polarizing lattices. In the context of the families Y_t and $X(a, b, d)$ above, the usual embedding

$$H \oplus E_8 \oplus E_8 \hookrightarrow H \oplus E_8 \oplus E_8 \oplus \langle -4 \rangle$$

corresponds to an algebraic parametrization

$$a(t) = (t + 16)(t + 256) , \quad b(t) = (t - 512)(t - 8)(t + 64) , \quad d(t) = 2^{12} 3^6 t^3$$

of a genus zero *modular curve*. To see the connection with classical modular curves, and indeed the Hodge-theoretic evidence for the underlying geometry, it is instructive to consider the Picard-Fuchs systems annihilating periods on the $K3$ surfaces involved.

Let $f(t)$ denote a period of the holomorphic 2-form on $X(a, b, d)$. The Griffiths-Dwork method for producing Picard-Fuchs systems yields (in an affine chart, where we have set $a = 1$)

$$\left(\frac{\partial^2}{\partial b^2} - 4d \frac{\partial^2}{\partial d^2} - 4 \frac{\partial}{\partial d} \right) f(b, d) = 0$$

and

$$\left((-1 + b^2 + d) \frac{\partial^2}{\partial b^2} + 2b \frac{\partial}{\partial b} + 4bd \frac{\partial^2}{\partial b \partial d} + 2d \frac{\partial}{\partial d} + \frac{5}{36} \right) f(b, d) = 0$$

[CDLW]. By reparametrizing in terms of variables j_1 and j_2

$$b^2 = \frac{(j_1 - 1)(j_2 - 1)}{j_1 j_2}, \quad d = \frac{1}{j_1 j_2}$$

we find that the Picard-Fuchs system completely decouples as

$$72j_1 \left(2(j_1 - 1)j_1 \frac{\partial^2}{\partial j_1^2} + (2j_1 - 1) \frac{\partial}{\partial j_1} \right) f(j_1, j_2) - 5f(j_1, j_2) = 0$$

and

$$72j_2 \left(2(j_2 - 1)j_2 \frac{\partial^2}{\partial j_2^2} + (2j_2 - 1) \frac{\partial}{\partial j_2} \right) f(j_1, j_2) - 5f(j_1, j_2) = 0.$$

This implies that the periods of the M -polarized $K3$ surfaces split naturally as products $f(j_1, j_2) = f_1(j_1) \cdot f_2(j_2)$.

At this point it is natural to ask whether the second order ordinary differential equation satisfied by $f(j)$ is itself a Picard-Fuchs equation for a family of elliptic curves. One can check for a family of elliptic curves over \mathbb{P}_t^1 in Weierstrass normal form

$$\{E_t\} := \{y^2 z - 4x^3 + g_2(t)xz^2 + g_3(t)z^3 = 0\} \subset \mathbb{P}^2$$

that the periods of a suitably normalized holomorphic one-form on E_t

$$g_2(t)^{\frac{1}{4}} \frac{dx}{y}$$

satisfy Picard-Fuchs equations of the form of the second order equations above. Thus, by the Hodge Conjecture, we expect there to be an algebraic correspondence between M -polarized $K3$ surfaces and abelian surfaces (with principal polarization) which split as a product of a pair of elliptic curves. This correspondence was made explicit in [C-D2]; we recall the necessary features for our higher K-theory computations in §6 below.

What then is the meaning of the special subfamily Y_t in terms of these split abelian surfaces? When specialized to the subfamily $Y_t = X(a(t), b(t), c(t))$, the Griffiths-Dwork method produces the following Picard-Fuchs differential equation

$$f^{(iii)}(t) + \frac{3(3t + 128)}{2t(t + 64)} f''(t) + \frac{13t + 256}{4t^2(t + 64)} f'(t) + \frac{1}{8t^2(t + 64)} f(t) = 0.$$

On a general parametrized disk in the moduli space \mathcal{M}_M , the Picard-Fuchs ODE will have rank 4, just as the full Picard-Fuchs system. The drop in rank indicates a special relationship between the two elliptic curves E_{τ_1} and E_{τ_2} corresponding to Y_t . A differential algebraic characterization of the curves in \mathcal{M}_M on which the Picard-Fuchs ODE drops in rank was given in [CDLW, Theorem 3.4]. In fact, in the M_2 -polarized case, the relationship is simply the existence of a two-isogeny between the two elliptic curves, i.e., $\tau_2 = 2 \cdot \tau_1$. More generally, the M_n -polarized case corresponds to a cyclic n -isogeny, i.e., $\tau_2 = n \cdot \tau_1$.

Given that M -polarized $K3$ surfaces correspond to abelian surfaces which are the products of a pair of elliptic curves, the natural modular parameters on the (rational) coarse moduli space \mathcal{M}_M are the elementary symmetric polynomials in the two j -invariants $j_1 = j(\tau_1)$ and $j_2 = j(\tau_2)$

$$\sigma := j_1 + j_2 \text{ and } \pi := j_1 \cdot j_2 .$$

In this notation, it is easy to identify explicit rational curves in \mathcal{M}_M over which the Picard-Fuchs differential equation has maximal rank ($= 4$). One such locus, which arises in the context of the construction of $K3$ surface fibered Calabi-Yau threefolds realizing hypergeometric variations, is specified by simply setting $\sigma = 1$ [No]. The Picard-Fuchs ODE has fourth order, and takes the following form

$$\begin{aligned} f^{(iv)}(s) + \frac{2(4s^2 - 3s - 2)}{s(s-1)(s+1)} f^{(iii)}(s) + \frac{1031s^3 - 553s^2 - 1175s - 167}{72s^2(s-1)(s+1)^2} f''(s) \\ + \frac{167s^2 - 239s - 118}{36s^2(s-1)(s+1)^2} f'(s) + \frac{385(s-1)^2}{20736s^4(s+1)^2} f(s) = 0 \end{aligned}$$

which splits as a tensor product of two very closely related factor second order ODEs

$$f_1''(s) + \frac{3s+1}{2s(s+1)} f_1'(s) + \frac{5}{144s(s+1)} f_1(s) = 0$$

and

$$f_2''(s) + \frac{3s+1}{2s(s+1)} f_2'(s) + \frac{5}{144s^2(s+1)} f_2(s) = 0$$

corresponding to the two families of elliptic curves satisfying $j_1(s) + j_2(s) = 1$. Examples such as this provide a source of families of explicit non-maximal families of $K3$ surfaces to explore.

Instead of looking at superlattices of $H \oplus E_8 \oplus E_8$ such as M_n , one can consider sublattices such as $N := H \oplus E_7 \oplus E_8$ and $S := H \oplus E_7 \oplus E_7$ [C-D3, C-D4]. Moduli spaces of $K3$ surfaces polarized by these sublattices are themselves parametrized by modular functions (and contain \mathcal{M}_M as a natural sublocus). For example, there is a normal form for N -polarized $K3$ surfaces extending the singular quartic normal form for M -polarized $K3$ surfaces with one additional monomial deformation

$$Q_N(a, b, c, d) : y^2zw - 4x^3z + 3axzw^2 + b zw^3 + cxz^2w - \frac{1}{2}(dz^2w^2 + w^4) = 0 .$$

The associated coarse moduli space \mathcal{M}_N is again an open subvariety of a weighted projective space

$$\mathcal{M}_N = \{[a, b, c, d] \in \mathbb{WP}(2, 3, 5, 6) \mid c \neq 0 \text{ or } d \neq 0\}$$

with modular parametrization

$$[a, b, c, d] = [\mathcal{E}_4, \mathcal{E}_6, 2^{12}3^5\mathcal{C}_{10}, 2^{12}3^6\mathcal{C}_{12}] ,$$

where \mathcal{E}_4 and \mathcal{E}_6 are genus-two Eisenstein series of weights 4 and 6, and \mathcal{C}_{10} and \mathcal{C}_{12} are Igusa's cusp forms of weights 10 and 12 [C-D3, Theorem 1.5].

The connection to genus two curve moduli here is suggestive of the fundamental geometric fact that N -polarized $K3$ surfaces are Shioda-Inose surfaces coming from principally-polarized abelian surfaces. The hypersurface normal form once again has two natural elliptic fibration structures with section, just as in the M -polarized case, and the Nikulin involution which gives rise to the Shioda-Inose structure can be seen most naturally as the operation of “translation by 2-torsion” in the

alternate elliptic fibration [C-D4]. There is a further extension to a normal form for S -polarized $K3$ surfaces. In this case, most of the related geometric structures are still present, and we find a still more general modular parametrization of \mathcal{M}_S . For all these families of lattice-polarized $K3$ surfaces in normal form, Picard-Fuchs equations can be obtained via the Griffiths-Dwork method applied directly to the singular quartic equations or in their realization as anticanonical hypersurfaces in Gorenstein toric Fano threefolds.

The explicit computations which follow in §6 and §7 offer a glimpse of the range of phenomena surrounding Theorem 1.1 which become accessible when we work with modular parametrizations of hypersurface normal forms for lattice polarized $K3$ surfaces equipped with well-chosen elliptic fibrations. Both generalization to related higher-dimensional moduli spaces and manipulation of the associated explicit Picard-Fuchs systems now becomes possible.

6. EXPLICIT K_1 CLASS ON A FAMILY OF SHIODA-INOSE $K3$ SURFACES

We now turn to a direct computation on the modular 2-parameter family $X_{a,b}$ of $M := H \oplus E_8 \oplus E_8$ -polarized (Picard-rank 18) $K3$'s introduced by Clingher and Doran [C-D2]. Here $X_{a,b}$ ($a, b \in \mathbb{C}$) is the minimal desingularization of

$$(57) \quad \left\{ Y^2 Z - P(\theta) W^2 Z - \frac{1}{2} Z^2 W - \frac{1}{2} W^3 = 0 \right\} \subset \mathbb{P}_{[Y:Z:W]}^2 \times \mathbb{P}_\theta^1,$$

where $P(\theta) := 4\theta^3 - 3a\theta - b$. The results of [C-L1] already tell us that the real regulator map

$$(58) \quad r_{2,1} : \mathrm{CH}^2(X_{a,b}, 1) \rightarrow \mathrm{Hom}_{\mathbb{R}}(H_v^{1,1}(X_{a,b}, \mathbb{R}), \mathbb{R})$$

is generically surjective, making $\Phi_{2,1}$ nontrivial for very general (a, b) . (We note that for those $X_{a,b}$ with Picard rank 18, $H_v^{1,1} = H_{tr}^{1,1}$.) The proof is based on *non-explicit* deformations of decomposable classes on Picard-rank 20 $K3$'s.

What we felt was missing here and in the literature are concrete indecomposable cycles on which $r_{2,1}$ and $\Phi_{2,1}$ are nontrivial, particularly those which arise naturally in the context of an internal elliptic fibration. In our example, the projection $X_{a,b} \rightarrow \mathbb{P}_\theta^1$ produces the so-called *alternate fibration* with 6 fibers of Kodaira type I_1 and one fiber of type I_{12}^* . The I_1 fibers provide *the* most natural source of classes in $\mathrm{CH}^2(X_{a,b}, 1)$ provided one can show their real regulators are nonzero.

This turns out to require some serious and interesting work, by first passing to a *Kummer* $K3$ family $K_{\alpha,\beta}$ which is the minimal resolution of both the quotient of $X_{a,b}$ by the Nikulin involution and the quotient of a product of elliptic curves $E_\alpha \times E_\beta$ by $(-\mathrm{id}, -\mathrm{id})$. This “intermediate” setting seems to be the one place where *both* the normalization of the rational curves supporting the family of K_1 classes (namely, a Néron 2-gon), and the closed $(1,1)$ -form against which we integrate its regulator current to compute $r_{2,1}$, are tractable. In fact, the form has some singularities, even after pulling back the rational curves, and so the computation requires careful additional justification.

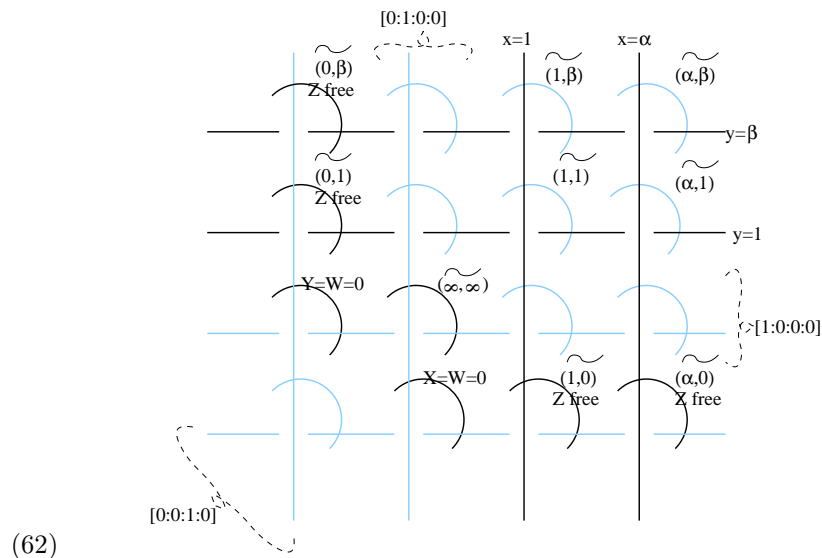
6.1. Kummer $K3$ geometry. We begin with a review of special features of the Kummer family from [C-D2], which has two parameters $\alpha, \beta \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$:

$$(59) \quad \check{K}'_{\alpha,\beta} := \{ Z^2 XY = (X - W)(X - \alpha W)(Y - W)(Y - \beta W) \} \subset \mathbb{P}^3$$

$$(60) \quad z^2xy = (x-1)(x-\alpha)(y-1)(y-\beta),$$
$$(61) \quad \{u^2 = x(x-1)(x-\alpha)\} \Longrightarrow : E_\alpha \curvearrowright J_\alpha : (x, u) \mapsto (x, -u)$$

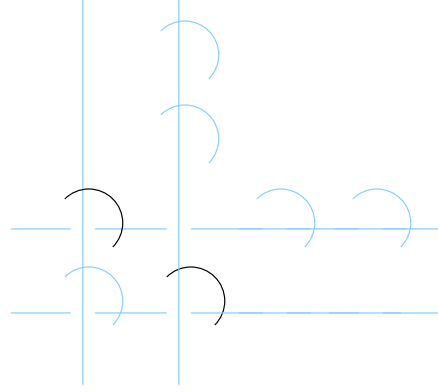
$$\{v^2 = y(y-1)(y-\beta)\} \Longrightarrow : E_\beta \curvearrowright J_\beta : (y, v) \mapsto (y, -v),$$

In the following diagram of rational curves on $K_{\alpha,\beta}$, the exceptional divisors are represented by arcs; while the proper transforms of the quotients of $E_\alpha \times \{2\text{-torsion point}\}$ resp. $\{2\text{-torsion point}\} \times E_\beta$ are represented by horizontal resp. vertical lines:



(Here “ (∞, ∞) ” stands for $\{W = 0, XY = Z^2\}$.) The projective model $\check{K}'_{\alpha, \beta}$ is the blow-down of $K_{\alpha, \beta}$ along the 13 rational curves depicted more faintly. Notice that

the configuration



(63)

has Dynkin diagram D_{10} , hence Kodaira type I_6^* .

We now describe an elliptic fibration of $K_{\alpha,\beta}$ which shall have:

- this I_6^* as its singular fiber at ∞ ;
- the lines $y = 1, y = \beta, x = 1, x = \alpha$ as sections;
- the lines marked $\widetilde{(1,0)}, \widetilde{(\alpha,0)}, \widetilde{(0,1)}, \widetilde{(0,\beta)}$ as bi-sections;
- the line marked (∞, ∞) as a 4-section; and
- 6 I_2 singular fibers, 4 of which have one of the lines marked $\widetilde{(1,\beta)}, \widetilde{(\alpha,\beta)}, \widetilde{(1,1)}$, or $\widetilde{(\alpha,1)}$ as one component.

Write

$$(64) \quad R(X, Y, W) := -\frac{X^2}{\alpha} - \frac{Y^2}{\beta} + \frac{\alpha+1}{\alpha}XW + \frac{\beta+1}{\beta}YW - W^2.$$

Then the fibration, which is really nothing but the pencil $|I_6^*|$, is given on the (singular) projective model by

$$(65) \quad \begin{array}{ccc} \check{K}'_{\alpha,\beta} & \longrightarrow & \mathbb{P}^1 \\ [X : Y : Z : W] & \longmapsto & [R(X, Y, W) : XY] =: [\mu : 1]. \end{array}$$

In either case, the smooth elliptic fibers \mathcal{E}_μ (resp. $\check{\mathcal{E}}'_\mu$) are double covers of the smooth conic curves

$$(66) \quad \mathcal{C}_\mu := \{R(X, Y, W) = \mu XY\} \subset \mathbb{P}^2,$$

branched over $(x, y) = (1, (1-\mu)\beta+1), (\alpha, (1-\mu\alpha)\beta+1), ((1-\mu)\alpha+1, 1), ((1-\mu\beta)\alpha+1, \beta)$. \mathcal{E}_μ is singular iff one of the following hold:

- $\mu = \infty$: then $\mathcal{E}_\infty = I_6^*$;
- $\mu \in \{1, \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\alpha\beta}\}$: then two of the branch points collide, making $\check{\mathcal{E}}'_\mu$ into an I_1 . \mathcal{E}_μ is then the (Kodaira type I_2) union of its proper transform with the exceptional divisor over the collision point – for example, for $\mu = 1$, $\mathcal{E}_1 = \check{\mathcal{E}}'_1 \cup \widetilde{(1,1)}$; or
- $\mu \in \left\{\frac{\alpha\beta+1}{\alpha\beta}, \frac{\alpha+\beta}{\alpha\beta}\right\}$: then the rational curve \mathcal{C}_μ acquires a node, so \mathcal{E}_μ has two nodes (again of type I_2).

This is all in case $J(E_\alpha) \neq J(E_\beta)$, i.e. $\beta \notin \left\{ \alpha, \frac{1}{\alpha}, 1 - \alpha, \frac{1}{1-\alpha}, \frac{\alpha}{\alpha-1}, \frac{\alpha-1}{\alpha} \right\}$. Below we will eventually specialize to the case $\beta = \alpha$, for which generically \mathcal{E}_1 is still an I_2 but $\mathcal{E}_{\frac{1}{\alpha}=\frac{1}{\beta}}$ becomes an I_4 .

6.2. Normalization of $\tilde{\mathcal{E}}'_1$. We will build our higher Chow cycle on \mathcal{E}_1 . One can see right away that it must have order-two monodromies about the components of $(\mathbb{P}^1 \times \{0, 1, \infty\}) \cup (\{0, 1, \infty\} \times \mathbb{P}^1)$, since the tangent vectors of the I_1 fiber $\tilde{\mathcal{E}}'_1$ at its singular point $(x, y, z) = (1, 1, 0)$ are $\left(1, -\frac{\beta}{\alpha}, \pm \sqrt{\frac{\beta}{\alpha}(1-\alpha)(1-\beta)}\right)$. Notice that with $\alpha = \beta$, the branches of the square root become single-valued hence the monodromy will disappear; this will have consequences later.

In order to compute, we need to parametrize $\tilde{\mathcal{E}}'_1$ by a \mathbb{P}^1 . The first step is to do this for \mathcal{C}_1 using stereographic projection. Putting $x = \Gamma + 1$, $y = \xi\Gamma + 1$ in its equation

$$\begin{aligned} 0 &= -\frac{x^2}{\alpha} - \frac{y^2}{\beta} + \frac{\alpha+1}{\alpha}x + \frac{\beta+1}{\beta}y - 1 - xy \\ (67) \quad &= \dots = -\left(\frac{1}{\alpha} + \frac{\xi^2}{\beta} + \xi\right)\Gamma^2 - \left(\frac{1}{\alpha} + \frac{\xi}{\beta}\right)\Gamma \end{aligned}$$

and solving for Γ , yields

$$(68) \quad (x(\xi), y(\xi)) = \left(\frac{\alpha\xi^2 + \alpha(\beta-1)\xi}{\Delta(\xi)}, \frac{\beta(\alpha-1)\xi + \beta}{\Delta(\xi)} \right),$$

where $\Delta(\xi) := \alpha\xi^2 + \alpha\beta\xi + \beta$.

The second step is to pull the affine equation of $\check{K}'_{\alpha,\beta}$ back along $\xi \mapsto (x(\xi), y(\xi))$ and again use an analogue of stereographic projection:

$$\begin{aligned} (69) \quad z^2 &= \frac{(x-1)(x-\alpha)(y-1)(y-\beta)}{xy} \\ &= \dots = \frac{(\alpha\xi+\beta)^2(\xi+\beta)(\alpha\xi+1)}{(\Delta(\xi))^2}. \end{aligned}$$

So the equation of the I_1 fiber $\tilde{\mathcal{E}}'_1$ is

$$(70) \quad (\Delta(\xi))^2 z^2 = (\xi + \beta)(1 + \alpha\xi)(\beta + \alpha\xi)^2,$$

which regarded as a curve in $\mathbb{P}^1_\xi \times \mathbb{P}^1_z$ has bidegree $(4, 2)$ and three nodes (hence of course genus 0). A curve of bidegree $(2, 1)$ must meet $\tilde{\mathcal{E}}'_1$ in 8 points with multiplicity; so taking it to pass through the nodes $\left(-\frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} - \frac{\beta}{\alpha}}, \infty\right)$, $\left(-\frac{\beta}{2} - \sqrt{\frac{\beta^2}{4} - \frac{\beta}{\alpha}}, \infty\right)$, $\left(-\frac{\beta}{\alpha}, 0\right)$ and the smooth point $(-\beta, 0)$, it must pass through one more point of $\tilde{\mathcal{E}}'_1$. Explicitly, these curves are of the form

$$(71) \quad \Delta(\xi)z = (\alpha\xi + \beta)(\xi + \beta)\gamma,$$

where $\gamma \in \mathbb{C}$ is a constant. To find the ξ -coordinate of the residual point we square RHS(71) and set equal to RHS(70), which yields

$$(72) \quad \xi(\gamma) = \frac{1 - \beta\gamma^2}{\gamma^2 - \alpha}.$$

Thinking of \mathbb{P}^1_γ as $\tilde{\mathcal{E}}'_1$ and \mathbb{P}^1_ξ as \mathcal{C}_1 , (72) gives the branched double cover $\tilde{\mathcal{E}}'_1 \twoheadrightarrow \mathcal{C}_1$, where the first map just identifies a pair of points – namely, those with

$\gamma^2 = \delta := \frac{\alpha\beta - \alpha}{\beta - \alpha\beta}$. The following table illustrates the relationship between functions on $\tilde{\mathcal{E}}'_1$:

γ^2	ξ	(x, y)
0	$-1/\alpha$	$(\alpha(1 - \beta) + 1, \beta)$
∞	$-\beta$	$(\alpha, \beta(1 - \alpha) + 1)$
δ	$-\beta/\alpha$	$(1, 1)$
$1/\beta$	0	$(0, 1)$
α	∞	$(1, 0)$
$-\alpha\beta + \alpha + 1$	$1 - \beta$	$(0, \beta)$
$\frac{1}{1 + \beta - \alpha\beta}$	$\frac{1}{1 - \alpha}$	$(\alpha, 0)$
roots of $\Delta(\xi(\gamma^2))$	roots of $\Delta(\xi)$	(∞, ∞)

The rows starting with 0 and ∞ correspond to the branch points of $\tilde{\mathcal{E}}'_1 \rightarrow \mathcal{C}_1$.

The third and last step is to find a coordinate \mathfrak{z} on $\tilde{\mathcal{E}}'_1 (\cong \mathbb{P}^1)$ which is 0 and ∞ (rather than $\pm\sqrt{\delta}$) at the two points mapping to the node of $\tilde{\mathcal{E}}'_1$, and ± 1 at the two branch points of $\tilde{\mathcal{E}}'_1 \rightarrow \mathcal{C}_1$. This is given by

$$(74) \quad \mathfrak{z} = \frac{\gamma + \sqrt{\delta}}{\gamma - \sqrt{\delta}} \longleftrightarrow \gamma = \sqrt{\delta} \frac{\mathfrak{z} + 1}{\mathfrak{z} - 1}.$$

Our higher Chow cycle in $CH^2(K_{\alpha, \beta}, 1)$ will then simply be

$$(75) \quad Z_{\alpha, \beta} := \left(\tilde{\mathcal{E}}'_1, \mathfrak{z} \right) + \left(\widetilde{(1, 1)}, g \right),$$

where g has zero and pole cancelling with those of \mathfrak{z} . (Note that while \mathfrak{z} is the “preferred” coordinate on the \mathbb{P}^1 , we will work mainly in γ below since this simplifies computations.) We remark that $Z_{\alpha, \beta}$ is defined as long as $\alpha, \beta \notin \{0, 1, \infty\}$ and $1 \notin \left\{ \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\alpha\beta}, \frac{\alpha\beta+1}{\alpha\beta}, \frac{\alpha+\beta}{\alpha\beta} \right\}$, but not quite well-defined: there is the issue of sign in $\mathfrak{z}^{\pm 1}$ (or equivalently, $\pm\sqrt{\delta}$) which leads to the predicted order-2 monodromies.

6.3. The (1, 1) current. On $E_\alpha \times E_\beta$ there is the closed, real-analytic (1, 1)-form

$$(76) \quad \omega = \frac{dx}{u} \wedge \overline{\left(\frac{dy}{v} \right)} = \frac{dx}{\sqrt{x(x-1)(x-\alpha)}} \wedge \overline{\left(\frac{dy}{\sqrt{y(y-1)(y-\beta)}} \right)},$$

and $\omega + \bar{\omega}$, $i(\omega - \bar{\omega})$ obviously span $H_{tr, \mathbb{R}}^{1,1}$. Clearly ω is invariant under $J_\alpha \times J_\beta$, hence is the pullback of a (1, 1)-current on $\tilde{K}_{\alpha, \beta}$, whose pullback ³ ω_K to $K_{\alpha, \beta}$ has integrable singularities transverse to the exceptional divisors: if locally the equation of one looks like $w = 0$, then there is a term of the form $\frac{dw \wedge d\bar{w}}{|w|}$. Now we could argue that this current ω_K is closed and represents a class in $H_{tr}^{1,1}(K_{\alpha, \beta}, \mathbb{C})$; but this approach runs into trouble because $(1, 1)$, where part of the cycle is supported, is an exceptional divisor. (The current’s singularity along this divisor makes the pairing “improper”, even though it “formally pulls back” to zero there.) Therefore,

³technically these observations should be expressed in terms of push-forwards, but the computations are better done as formal pullbacks.

we will simply carry out an *ad hoc* pairing between ${}^4 r_{2,1}(Z_{\alpha,\beta})$ and ω_K on $\tilde{\mathcal{E}}'_1$, then interpret it on $E_\alpha \times E_\beta$ where ω is smooth.

So taking ι_1 denote the inclusion $\tilde{\mathcal{E}}'_1 \hookrightarrow K_{\alpha,\beta}$, we must compute $\iota_1^* \omega_K$. This is done by “formally” pulling back the above form (76) under $\xi \mapsto (x(\xi), y(\xi))$: after some calculation, we obtain

$$(77) \quad \frac{-(\alpha\xi^2 + 2\beta\xi + \beta(\beta-1))(\alpha(\alpha-1)\xi^2 + 2\alpha\xi + \beta) d\xi \wedge d\bar{\xi}}{|\Delta(\xi)||\xi||\alpha\xi + \beta||\xi + (\beta-1)||(\alpha-1)\xi + 1|\sqrt{(\xi + \beta)(\alpha\xi + 1)}},$$

a sort of multivalued form on \mathcal{C}_1 . Pulling this back (again “formally”) to $\tilde{\mathcal{E}}'_1 \cong \mathbb{P}_\gamma^1$ via $\gamma \mapsto \xi(\gamma)$ then yields (with apologies to the reader) $\iota_1^* \omega_K =$

$$(78) \quad \frac{-4|\alpha\beta-1|}{|\beta||1-\alpha|} \cdot \frac{\{(\alpha\beta^2-\beta^2-\beta)\gamma^4+2\beta\gamma^2+(\alpha^2\beta^2-\alpha^2\beta+\alpha-2\alpha\beta)\}\gamma d\gamma}{|\gamma^2-\alpha||1-\beta\gamma^2||\gamma^2-\delta||\gamma^2-(1+\alpha-\alpha\beta)|} \wedge$$

$$\frac{\{(\alpha^2\beta^2-\alpha\beta^2+\beta-2\alpha\beta)\gamma^4+2\alpha\gamma^2+(\alpha^2\beta-\alpha^2-\alpha)\}d\bar{\gamma}}{|(1+\beta-\alpha\beta)\gamma^2-1||\beta\gamma^4+(\alpha^2\beta^2-3\alpha\beta)\gamma^2+\alpha|}.$$

While complicated, the 14 poles of this $(1,1)$ current are all of the integrable form mentioned above, and their locations are precisely the points where $\tilde{\mathcal{E}}'_1$ hits the exceptional divisors: $(1,1)$, $(1,0)$, $(\alpha,0)$, $(0,1)$, $(0,\beta)$ twice each; (∞,∞) four times.

Along the locus $\alpha = \beta$, this form simplifies a little: $\iota_1^* \omega_K =$

$$(79) \quad -4|\alpha+1| \cdot \frac{\{(\alpha^2-\alpha-1)\gamma^4+2\gamma^2+(\alpha^3-\alpha^2-2\alpha+1)\}\gamma d\gamma}{|\gamma^2-\alpha||1-\alpha\gamma^2||\gamma^2+1||\gamma^2-(1+\alpha-\alpha^2)|} \wedge$$

$$\frac{\{(\alpha^3-\alpha^2-2\alpha+1)\gamma^4+2\gamma^2+(\alpha^2-\alpha-1)\}d\bar{\gamma}}{|(1+\alpha-\alpha^2)\gamma^2-1||\gamma^4+(\alpha^3-3\alpha)\gamma^2+1|}.$$

6.4. The pairing. The next step is simply to integrate $\log|\mathfrak{z}|$ against $\iota_1^* \omega_K$ on $\tilde{\mathcal{E}}'_1$.

As $\log|\mathfrak{z}| = \log\left|\frac{\gamma+\sqrt{\delta}}{\gamma-\sqrt{\delta}}\right|$, this integral will have a multivalued behavior as indicated above. It is singular but absolutely convergent: the worst behavior is at $\gamma = \pm\sqrt{\delta}$ where it locally takes the form $\int_{D_\epsilon} \frac{\log|z|}{|z|} dz \wedge d\bar{z}$, which is equivalent to $\int_0^\epsilon (\log r) dr$.

But setting $\alpha = \beta$ ($\implies \delta = -1$) kills this monodromy, allowing for a well-defined choice of $Z_{\alpha,\alpha} \in CH^2(X_{a,b},1)$ over $\mathbb{P}^1 \setminus \{0,1,\infty,-1,2\}$ (see the end of §3.2). On a smooth compactification of the total space $\mathcal{X} \xrightarrow{\rho} \mathbb{P}_\alpha^1$, the “total cycle” is easily seen to have residues (i.e. $\log|\mathfrak{z}|$ blows up) along $X_{-1,-1}$ *only*. By the localization sequence for higher Chow groups, it can in fact be extended to all of $\rho^{-1}(\mathbb{P}^1 \setminus \{-1\})$. Most importantly, eliminating the monodromy makes the integrals

$$(80) \quad \psi(\alpha) = \int_{\mathbb{P}^1} \log\left|\frac{\gamma+i}{\gamma-i}\right| \Re(\iota_1^* \omega_K), \quad \eta(\alpha) = \int_{\mathbb{P}^1} \log\left|\frac{\gamma+i}{\gamma-i}\right| \Im(\iota_1^* \omega_K)$$

real-analytic functions of $\alpha \in \mathbb{P}^1 \setminus \{0,1,\infty,-1\}$.

Now on $E_\alpha \times E_\alpha$, by considering pullbacks to the diagonal, one sees immediately that $i(\omega - \bar{\omega})$ is the algebraic class whilst $\omega + \bar{\omega}$ is the transcendental one. Clearly the same story holds on $K_{\alpha,\alpha}$. So to check generic indecomposability of $Z_{\alpha,\alpha}$ we need to demonstrate that $\psi(\alpha)$ (rather than $\eta(\alpha)$) is generically nonzero. Clearly it will suffice to show that $\lim_{\alpha \rightarrow 1} \psi(\alpha) \neq 0$.

⁴pairing the regulator with $\omega_K + \bar{\omega}_K$ and $i(\omega_K - \bar{\omega}_K)$ to get two real numbers, is equivalent to pairing it with ω_K to get a single complex number.

Setting $\alpha = 1$ in (79) yields

$$(81) \quad \begin{aligned} \iota_1^* \omega_K &= \frac{-8|\gamma^2-1|^4 \gamma d\gamma \wedge d\bar{\gamma}}{|\gamma^2-1|^6 |\gamma^2+1| |2\gamma^2-1|} \\ &= \frac{-8\gamma d\gamma \wedge d\bar{\gamma}}{|\gamma^2+1| |\gamma^2-1| |2\gamma^2-1|} = \frac{16r \{i \cos \theta - \sin \theta\} dx \wedge dy}{|\gamma^2+1| |\gamma^2-1| |2\gamma^2-1|}, \end{aligned}$$

where $\gamma = x + iy = re^{i\theta}$. Because of the cancellations in the second step, it requires some analysis to prove that $\int_{\mathbb{P}^1} \log |\mathfrak{z}| \Re(\iota_1^* \omega_K)$ at $\alpha = 1$ actually computes the limit of ψ . This is done in the appendix to this section, and so we have

$$(82) \quad -\frac{1}{16} \lim_{\alpha \rightarrow 1} \psi(\alpha) = \int_{\mathbb{P}^1} \frac{\log \left| \frac{\gamma+i}{\gamma-i} \right| \cdot \sin \theta}{|\gamma^2+1| |\gamma^2-1| |2\gamma^2-1|} dx \wedge dy.$$

Now simply notice that

- the integral over \mathbb{P}^1 in (82) is double that over the upper half plane, since $\log \left| \frac{\gamma+i}{\gamma-i} \right|$ and $\sin \theta$ are both odd in γ ; and
- the integrand is (where nonsingular) strictly positive on the upper half plane.

We conclude that (82) is a positive real number, finishing this part of the argument.

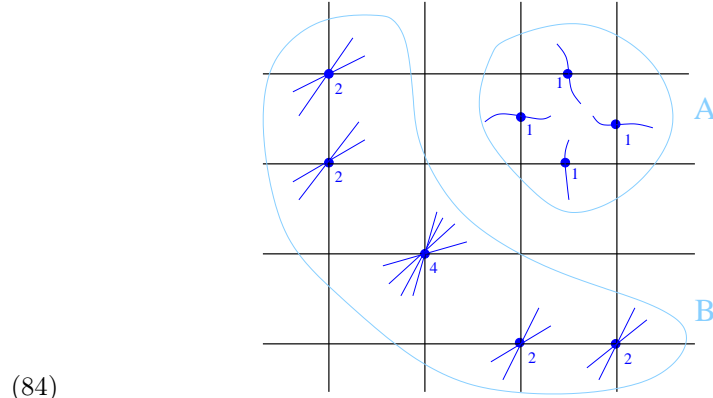
Remark 6.1. It is more natural to normalize ω_K , and hence ψ , by dividing out by $\left| \int_{E_\alpha} \frac{dx}{y} \wedge \overline{\left(\frac{dx}{y} \right)} \right|$. One can show – either using formula (80) or from general principles to be explained in [Ke] – that this modified ψ is asymptotic to $C \cdot \log |\alpha + 1|$ as $\alpha \rightarrow -1$, and goes to zero as $\alpha \rightarrow 0, 1, \infty$. We don't need this here.

6.5. Interpretation of the integrals. From the generic nontriviality of $\psi(\alpha)$, we know that

$$(83) \quad \int_{\widetilde{\mathcal{E}}'_1} (\log |\mathfrak{z}|) \iota_1^* \omega_K$$

is nonzero for generic α, β . We will show that this integral has meaning as an invariant of $Z_{\alpha, \beta}$ in roundabout fashion, by first exhibiting it as an invariant of a related cycle on $E_\alpha \times E_\beta$.

For generic μ , the image $\check{\mathcal{E}}_\mu$ of $\widetilde{\mathcal{E}}'_\mu$ in $\check{K}_{\alpha, \beta}$ is a curve with intersection numbers as follows:



where the horizontal and vertical lines have the same meaning as in the earlier picture (62). Obviously its normalization is elliptic, with 4 smooth branch points over the conic \mathcal{C}_μ at the points of type (A). Its preimage \mathcal{D}_μ in $E_\alpha \times E_\beta$ is an

irreducible curve with singularities at the points of type **(B)**; and its normalization can be thought of as a double cover of the normalization of $\tilde{\mathcal{E}}_\mu$, branched at the points lying over these singularities. An easy Riemann-Hurwitz calculation shows that $\tilde{\mathcal{D}}_\mu$ has genus 7.

As $\mu \rightarrow 1$, \mathcal{D}_μ and $\tilde{\mathcal{E}}_\mu$ each acquire a new node, one mapping to the other: $\mathcal{O} \mapsto (1, 1)$. The local description (at the nodes) of the map $\mathcal{D}_1 \rightarrow \tilde{\mathcal{E}}_1$ is “ $z \mapsto z^2$ ” on each branch separately. (Note that $\tilde{\mathcal{D}}_1$ has genus 6.) Therefore, the pullback $\tilde{\mathfrak{z}} \in \mathbb{C}(\tilde{\mathcal{D}}_1)^*$ of the function \mathfrak{z} on $\tilde{\mathcal{E}}_1$ pushes forward to \mathcal{D}_1 to yield a K_1 -class: its double-zero and double-pole cancel at \mathcal{O} . Further, the real regulator current $\log |\tilde{\mathfrak{z}}| \delta_{\mathcal{D}_1}$ pairs against $\omega \in \Gamma(E_\alpha \times E_\beta, A^{1,1})_{d\text{-closed}}$ to yield

- (a) an honest invariant of this K_1 -class; and
- (b) twice the value of the integral (83), since ω and $\tilde{\mathfrak{z}}$ are both invariant under the involution flipping $\tilde{\mathcal{D}}_1$ over $\tilde{\mathcal{E}}_1$.

Consider the diagram

$$(85) \quad \begin{array}{ccccc} & \tilde{K}_{\alpha,\beta} & & \tilde{K}'_{\alpha,\beta} & \\ & \swarrow \pi_2 & \searrow \pi_1 & \swarrow \pi'_1 & \searrow \pi'_2 \\ E_\alpha \times E_\beta & & K_{\alpha,\beta} & & X_{a,b} \\ & \searrow 2:1 & \downarrow & \swarrow 2:1 & \\ & \tilde{K}_{\alpha,\beta} & \tilde{K}'_{\alpha,\beta} & \tilde{K}''_{\alpha,\beta} & \end{array}$$

in which $X_{a,b}$ is the Shioda-Inose $K3$, $\tilde{K}''_{\alpha,\beta}$ its quotient by the Nikulin involution, and the relationship between the two sets of parameters is given by

$$(86) \quad J(E_\alpha) + J(E_\beta) = a^3 - b^2 + 1, \quad J(E_\alpha) \cdot J(E_\beta) = a^3.$$

The preimage of \mathcal{D}_1 under π_2 consists of $\tilde{\mathcal{D}}_1$ and \mathcal{W} (an exceptional \mathbb{P}^1 with coordinate “ w ”) meeting at $w = 0$ and $w = \infty$ on \mathcal{W} . The map π_1 pushes this down to $\mathcal{E}_1 = \tilde{\mathcal{E}}_1 \cup (1, 1)$, where the map from \mathcal{W} to $(1, 1)$ is given by $w \mapsto w^2$. Setting

$$(87) \quad \tilde{Z}_{\alpha,\beta} := (\tilde{\mathcal{D}}_1, \tilde{\mathfrak{z}}) + (\mathcal{W}, w^2) \in CH^2(\tilde{K}_{\alpha,\beta}, 1),$$

we have $\pi_{1,*}(\tilde{Z}_{\alpha,\beta}) = 2Z_{\alpha,\beta}$ and $\pi_{2,*}(\tilde{Z}_{\alpha,\beta}) = \pi_{2,*}(\tilde{\mathcal{D}}_1, \tilde{\mathfrak{z}})$. By (a), (b), and functoriality of $r_{2,1}$, it now follows that the pairing (83) indeed computes the regulator of $Z_{\alpha,\beta}$.

What about cycles on $X_{a,b}$? The 2:1 birational correspondence provided by π'_1 and π'_2 identify its alternate fibration with the elliptic fibration of $K_{\alpha,\beta}$ (generically in 2:1 étale fashion). More precisely, we have a diagram

$$(88) \quad \begin{array}{ccc} K_{\alpha,\beta} & \xleftarrow{2:1} & X_{a,b} \\ \downarrow & & \downarrow \\ \mathbb{P}_\mu^1 & \xleftarrow{\quad} & \mathbb{P}_\theta^1 \end{array}$$

where the bottom map is of the form $\theta \mapsto q\theta + p$ (with p and q constants dependent on α, β). On $\tilde{K}'_{\alpha,\beta}$ there is a K_1 -cycle $\tilde{Z}'_{\alpha,\beta}$ supported on an I_2 , $\pi'_{1,*}$ of which is

$2Z_{\alpha,\beta}$, and a similar analysis goes through, proving indecomposability of the I_1 -supported $\mathcal{Z}_{a,b} := \pi'_{2,*}(\tilde{Z}'_{\alpha,\beta})$. So our integral (83) computes the real regulator of a trio of explicit cycles, on $E_\alpha \times E_\beta$, $K_{\alpha,\beta}$, and $X_{a,b}$.

Appendix to Section 3. Here we perform the analytic estimate which establishes the limiting assertion in §3.4, for $\alpha \rightarrow 1$. It will suffice to consider the behavior of the integral in a fixed neighborhood of one of the points (we use $\gamma = +1$) where zeroes and poles collide. Write $\chi = \alpha - 1$, $\gamma^2 = \zeta + 1$, and let $D_r(c)$ denote the open disk about c of radius r .

We may leave out the polynomial factors with no zero or pole approaching $\zeta = 0$, and approximate the locations of zeroes and poles to the lowest order required to distinguish them. The problem is then to show that

$$(89) \quad \int_{|\zeta| < \frac{1}{2}} \frac{(\zeta - 3\chi)(\overline{\zeta + 3\chi})(\zeta + \chi)(\overline{\zeta - \chi}) \log |\mathfrak{z}| d\zeta \wedge d\bar{\zeta}}{|\zeta - (\chi + \chi^2)| |\zeta - (\chi - \chi^2)| |\zeta + (\chi + \chi^2)| |\zeta + (\chi - \chi^2)| |\zeta - i\sqrt{3}\chi| |\zeta + i\sqrt{3}\chi|}$$

limits to

$$(90) \quad \int_{|\zeta| < \frac{1}{2}} \log |\mathfrak{z}| \frac{d\zeta \wedge d\bar{\zeta}}{|\zeta|^2}$$

as $\chi \rightarrow 0^+$ along the real axis. Given $\epsilon > 0$, and taking $0 < \chi < \epsilon/3$, it is obvious that the integrand in (89) converges uniformly on $\epsilon < |\zeta| < 1/2$. We claim that the remaining part $\int_{|\zeta| < \epsilon}$ of the integral, independently of $\chi \in (0, \frac{\epsilon}{3})$, is bounded by $1000\pi\epsilon$. This will prove the desired convergence.

To verify the claim, we first remark that $\log |\mathfrak{z}|$ is zero for all $\zeta \in \mathbb{P}^1(\mathbb{R})$; in fact, we shall just use that $|\log |\mathfrak{z}|| < |\zeta|$. Next, note that on the complement in $D_\epsilon(0)$ of the four disks $D_{\frac{\chi}{2}}(\chi)$, $D_{\frac{\chi}{2}}(-\chi)$, $D_{\frac{\chi}{2}}(i\sqrt{3}\chi)$, $D_{\frac{\chi}{2}}(-i\sqrt{3}\chi)$,

$$(91) \quad \frac{|\zeta + 3\chi| |\zeta + \chi|}{|\zeta + \chi + \chi^2| |\zeta + \chi - \chi^2|} = \frac{|\lambda + 2\chi| |\lambda|}{|\lambda + \chi^2| |\lambda - \chi^2|} = \frac{|1 + \frac{2\chi}{\lambda}|}{|1 + \frac{\chi^2}{\lambda}| |1 - \frac{\chi^2}{\lambda}|}$$

(where $\lambda := \zeta + \chi$) is bounded by 6, since $|\frac{2\chi}{\lambda}| \leq 4$, $|\frac{\chi^2}{\lambda}| \leq 2\chi$ and we are assuming χ is small. The same is true for $\frac{|\zeta - 3\chi| |\zeta - \chi|}{|\zeta - \chi + \chi^2| |\zeta - \chi - \chi^2|}$; and similarly, $\frac{|\zeta|^2}{|\zeta - i\sqrt{3}\chi| |\zeta + i\sqrt{3}\chi|}$ is bounded by 9. So the integral over $D_\epsilon(0) \setminus \{4 \text{ disks}\}$ is bounded by

$$(92) \quad \int_{|\zeta| < \epsilon} 9 \cdot 6^2 \cdot \frac{|d\zeta \wedge d\bar{\zeta}|}{|\zeta|} = 324 \cdot 2\pi \int_0^\epsilon \frac{r dr}{r} < 650\pi\epsilon.$$

Now consider (say) the right half of $D_{\frac{\chi}{2}}(-\chi)$: here the absolute value of the integrand, apart from the $\frac{1}{|\lambda - \chi^2|}$, is

$$(93) \quad \frac{|\lambda - 4\chi| |\lambda - 2\chi|}{|\lambda - (2\chi + \chi^2)| |\lambda - (2\chi - \chi^2)|} \cdot \frac{|\lambda|}{|\lambda + \chi^2|} \cdot \frac{|\lambda + 2\chi| |\lambda - \chi|}{|\lambda - i\sqrt{3}\chi| |\lambda + i\sqrt{3}\chi|} \leq 6 \cdot 1 \cdot \frac{10}{3} \leq 20.$$

We have then

$$(94) \quad 20 \int_{D_{\frac{\chi}{2}}(0) \cap \Re(\lambda) > 0} \frac{|d\lambda \wedge d\bar{\lambda}|}{|\lambda - \chi^2|} \leq 20 \int_{D_\chi(0)} \frac{|d\lambda \wedge d\bar{\lambda}|}{|\lambda|} = 40\pi\chi < \frac{40}{3}\pi\epsilon,$$

together with similar estimates on 3 other half-disks. The estimates for $D_{\frac{x}{2}}(\pm i\sqrt{3}\chi)$ are each $\frac{250}{3}\pi\epsilon$. Adding everything from inside and outside the 4 disks, we are safely under $1000\pi\epsilon$.

7. THE TRANSCENDENTAL REGULATOR FOR A PICARD RANK 20 $K3$

Here we specialize to the case (cf. §3.5)

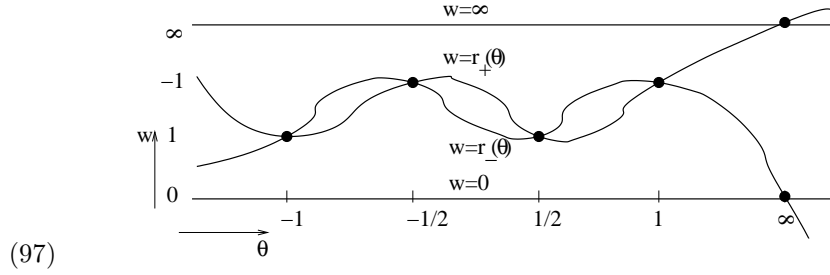
$$(95) \quad \alpha = \frac{1}{2} = \beta, \quad a = 1, b = 0,$$

in which case $E_\alpha, E_\beta \cong \mathbb{C}/\mathbb{Z}(1, i)$ are CM and $p = 3, q = -2$ (cf. [C-D2]). The singular fibres are at $\theta = \pm\frac{1}{2}$ (type I_2) and ± 1 (type I_1) in $X := X_{1,0}$, and at $\mu = 2, 4$ (type I_4) and $1, 5$ (type I_2) in $K_{\frac{1}{2}, \frac{1}{2}}$. Recalling that our original cycle was supported over $\mu = 1$, which in this specialization has *remained* an I_2 fiber (hence preserving the cycle), its transform $\mathcal{Z} := \mathcal{Z}_{1,0}$ is supported over $\theta = 1$ in X .

To take a closer look at the fibration structure of X , we use its affine equation

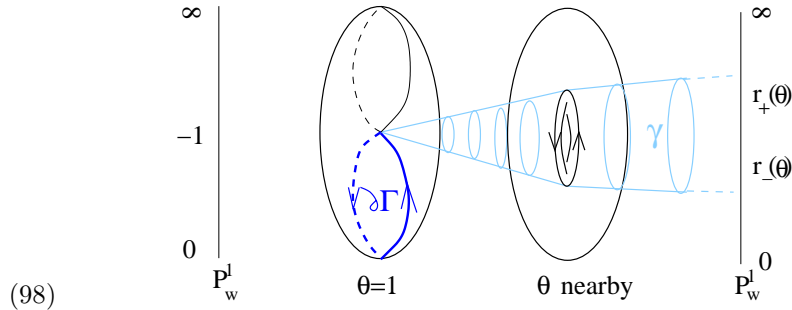
$$(96) \quad 2y^2 = w(w^2 + 2\underbrace{\{4\theta^3 - 3\theta\}}_{=: Q_\theta(w)}w + 1)$$

to sketch the families of branch points of the elliptic fibers:



Here $r_\pm(\theta)$ are the roots of $Q_\theta(w)$, which are both negative real for $\theta \in [1, \infty)$, with $r_- = r_+^{-1}$. For purposes of constructing transcendental cycles, one should imagine all the branch points coalescing at $\theta = \infty$ since that fiber, an I_{12}^* , has trivial H_1 .

In particular, considering the fiber over $\theta = 1$, the membrane Γ we use for the transcendental regulator computation must bound on the indicated cycle $\partial\Gamma = T_{\mathcal{Z}}$:



which is a double cover of the path $[-1, 0] \subset \mathbb{P}_w^1$. The transcendental 2-cycle γ is the family of double covers of $[r_-, r_+]$ as θ goes from 1 to ∞ .

By basic residue theory the holomorphic $(2, 0)$ form on X is given by

$$(99) \quad \omega_0 = \frac{dw \wedge d\theta}{y}$$

in the affine coordinates. If

$$(100) \quad \int_{\gamma} \omega_0 = 2\sqrt{2} \int_{\theta=1}^{\infty} \left(\int_{r_-(\theta)}^{r_+(\theta)} \frac{dw}{\sqrt{wQ_{\theta}(w)}} \right) d\theta \quad (> 0)$$

is one transcendental period, then using the automorphism $j : X \rightarrow X$ given by $(w, y, \theta) \mapsto (-w, -iy, -\theta)$, we have

$$(101) \quad \int_{j(\gamma)} \omega_0 = \int_{\gamma} j^* \omega_0 = i \int_{\gamma} \omega_0.$$

Normalizing ω_0 to $\omega := \frac{\omega_0}{\int_{\gamma} i\omega_0}$, we find that $\Phi_{2,1}$ is described by

$$(102) \quad \begin{array}{ccc} CH^2(X, 1) & \longrightarrow & \mathbb{C}/\mathbb{Z}[i] \\ \mathcal{Z} & \longmapsto & \int_{\Gamma} \omega \end{array},$$

which for our particular cycle is

$$(103) \quad \begin{aligned} \kappa &:= \int_{\Gamma} \omega = 2 \int_{\theta=1}^{\infty} \int_{w=r_+(\theta)}^0 \omega \\ &= \frac{\int_1^{\infty} \int_{r_+(\theta)}^0 \frac{dw}{\sqrt{-wQ_{\theta}(w)}} d\theta}{\int_1^{\infty} \int_{r_-(\theta)}^{r_+(\theta)} \frac{dw}{\sqrt{wQ_{\theta}(w)}} d\theta} \in \mathbb{R}_+. \end{aligned}$$

That is, *the nontriviality of $\Phi_{2,1}(\mathcal{Z})_{\mathbb{Q}}$ is equivalent to irrationality of κ .*

The situation is highly reminiscent of a computation by Harris [Ha] of the Abel-Jacobi map for the Ceresa cycle of the Fermat quartic curve. In that case, a computer computation suggested that the comparable invariant $\kappa' \in \mathbb{R}/\mathbb{Q}$ was nontrivial. This would have implied that the cycle was nontorsion modulo rational equivalence, a fact later proved by Bloch [B2] using his ℓ -adic AJ map. Since the Fermat Jacobian is defined over $\bar{\mathbb{Q}}$, the Bloch-Beilinson conjecture predicts injectivity of the usual AJ map, and hence the irrationality of κ' . One might, in conclusion, speculate that a similar story unfolds here.

REFERENCES

- [As] M. Asakura, *On dlog image of K_2 of elliptic surface minus singular fibers*, preprint, 2006, arXiv:math/0511190v4.
- [Be] A. Beilinson, *Higher regulators of modular curves*, in “Applications of algebraic K -theory to algebraic geometry and number theory (Boulder, CO, 1983)”, Comtemp. Math. 55, AMS, Providence, RI, 1986, 1-34.
- [B] S. Bloch, *Algebraic cycles and higher K -theory*, Adv. Math. **61**, 1986, 267 - 304.
- [B2] S. Bloch, *Algebraic cycles and values of L -functions*, J. Reine Angew. Math. 350 (1984), 94-108.
- [C-MS-P] J. Carlson, S. Müller-Stach, C. Peters, *Period Mappings and Period Domains*, Cambridge Studies in Advanced Mathematics **85**, Cambridge University Press (2003).
- [C-L1] X. Chen and J. D. Lewis, . *The Hodge- \mathcal{D} -conjecture for $K3$ and Abelian surfaces*, J. Algebraic Geom. **14** (2005), no. 2, 213-240.
- [C-L2] X. Chen and J. D. Lewis, *Real regulators on self-products of $K3$ surfaces*, J. Algebraic Geom. **20** (2011), no. 1, 101 - 125.
- [C-L3] X. Chen and J. D. Lewis, *Transcendental regulators on self-products of $K3$ surfaces*. Preprint, 2011.
- [C-D1] A. Clingher and C. F. Doran, *On $K3$ surfaces with large complex structure*, Adv. Math. 215 (2007).
- [C-D2] A. Clingher and C. F. Doran, *Modular invariants for lattice polarized $K3$ surfaces*, Mich. Math. J. 55 (no. 2) (2007), 355-393.
- [C-D3] A. Clingher and C. F. Doran, *Lattice polarized $K3$ surfaces and Siegel modular forms*, arXiv:1004.3503v1 (2010).

- [C-D4] A. Clingher and C. F. Doran, *Note on a geometric isogeny of K3 surfaces*, International Mathematics Research Notices, DOI:10.1093/imrn/rnq230 (2010).
- [CDLW] A. Clingher, C. Doran, J. M. Lewis, U. Whitcher, *Normal forms, K3 surface moduli, and modular parametrizations*, Groups and Symmetries: From Neolithic Scots to John McKay (J. Harnad and P. Winternitz, eds.), CRM Proceedings and Lecture Notes, Vol. 47 (2009), 81-98.
- [Dol] I. V. Dolgachev, *Mirror symmetry for lattice polarized K3 surfaces*, J. Math. Sci. 81 (1996), no. 3, 2599-2630.
- [Dor1] C. F. Doran, *Picard-Fuchs uniformization: modularity of the mirror map and mirror moonshine*, The Arithmetic and Geometry of Algebraic Cycles (Banff, 1998) (B. B. Gordon, J. D. Lewis, S. Muller-Stach, S. Saito, and N. Yui, eds.), CRM Proceedings and Lecture Notes, Vol. 24, (2000), 257-281.
- [Dor2] C. F. Doran, *Picard-Fuchs uniformization and modularity of the mirror map*, Comm. Math. Phys. 212, (2000), no. 3, 625-647.
- [G-L] B. Gordon and J. D. Lewis, *Indecomposable higher Chow cycles on products of elliptic curves*, J. Alg. Geom. 8 (1999), 543-567.
- [Gr] P. A. Griffiths, *A theorem concerning the differential equations satisfied by normal functions associated to algebraic cycles*, Amer. J. Math. **101** (1979) no. 1, 94-131.
- [Ha] B. Harris, *Homological versus algebraic equivalence in a Jacobian*, Proc. Nat. Acad. Sci. U.S.A. 80 (1983), no. 4 i., 1157-1158.
- [Ke] M. Kerr, *Indecomposable K_1 classes on elliptic K3 surfaces*, in preparation.
- [K-L] M. Kerr, J. D. Lewis, *The Abel-Jacobi map for higher Chow groups, II*, Invent. Math. **170**, (2007), no. 2, 355-420.
- [KLM] M. Kerr, J. D. Lewis, S. Mller-Stach, *The Abel-Jacobi map for higher Chow groups*, Compositio Math. **142**, (2006), 374 - 396.
- [L] J. D. Lewis, *A note on indecomposable motivic cohomology classes*, J. reine angew. Math. **485** (1997), 161 - 172.
- [MS] S. Mller-Stach, *Constructing indecomposable motivic cohomology classes on algebraic surfaces*, J. Algebraic Geometry **6**, 1997, 513 - 543.
- [No] A. Novoseltsev, *Calabi-Yau hypersurfaces and complete intersections in toric varieties*, University of Alberta Ph.D. Thesis, 2011.
- [PL-MS] Pedro Luis del Angel, Stefan J. Mller-Stach, *The transcendental part of the regulator map for K_1 on a mirror family of K3-surfaces*, Duke Math. J. **112** (2002), no. 3, 581-598.
- [Sa] M. Saito, *Direct image of logarithmic complexes and infinitesimal invariants of cycles*, in Algebraic Cycles and Motives, Vol. 2, Edited by J. Nagel and C. Peters, London Mathematical Society Lecture Notes Series **344**, Cambridge University Press, (2007), 304 - 318.
- [T] I. U. Turkmen, *Regulator Indecomposable Cycles on a Product of Elliptic Curves*. Preprint, 2011.
- [Wh] U. Whitcher, *Polarized families of K3 hypersurfaces*, University of Washington Ph.D. Thesis, 2009.
- [Zi] R. Zigmund, *Triangle configurations and Beilinson's conjecture for $K_1^{(2)}$ of the product of a curve with itself*, Durham University Ph.D. Thesis, 2010.

632 CENTRAL ACADEMIC BUILDING, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA T6G
2G1, CANADA

E-mail address: `xichen@math.ualberta.ca`

632 CENTRAL ACADEMIC BUILDING, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA T6G
2G1, CANADA

E-mail address: `doran@math.ualberta.ca`

DEPARTMENT OF MATHEMATICS, CAMPUS BOX 1146, WASHINGTON UNIVERSITY IN ST. LOUIS,
ST. LOUIS, MO 63130, USA

E-mail address: `matkerr@math.wustl.edu`

632 CENTRAL ACADEMIC BUILDING, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA T6G
2G1, CANADA

E-mail address: `lewisjd@ualberta.ca`